K.T. Tang

Mathematical Methods for Engineers and Scientists 1

Complex Analysis, Determinants and Matrices

Springer

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With 49 Figures and 2 Tables

Springer

Professor Dr. Kwong-Tin Tang
Pacific Lutheran University
Department of Physics
Tacoma, WA 98447, USA
E-mail: tangka@plu.edu

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Preface

For some 30 years, I have taught two "Mathematical Physics" courses. One of them was previously named "Engineering Analysis." There are several textbooks of unquestionable merit for such courses, but I could not find one that fitted our needs. It seemed to me that students might have an easier time if some changes were made in these books. I ended up using class notes. Actually, I felt the same about my own notes, so they got changed again and again. Throughout the years, many students and colleagues have urged me to publish them. I resisted until now, because the topics were not new and I was not sure that my way of presenting them was really much better than others. In recent years, some former students came back to tell me that they still found my notes useful and looked at them from time to time. The fact that they always singled out these courses, among many others I have taught, made me think that besides being kind, they might even mean it. Perhaps it is worthwhile to share these notes with a wider audience.

It took far more work than expected to transcribe the lecture notes into printed pages. The notes were written in an abbreviated way without much explanation between any two equations, because I was supposed to supply the missing links in person. How much detail I would go into depended on the reaction of the students. Now without them in front of me, I had to decide the appropriate amount of derivation to be included. I chose to err on the side of too much detail rather than too little. As a result, the derivation does not look very elegant, but I also hope it does not leave any gap in students' comprehension.

Precisely stated and elegantly proved theorems looked great to me when I was a young faculty member. But in later years, I found that elegance in the eyes of the teacher might be stumbling blocks for students. Now I am convinced that before the student can use a mathematical theorem with confidence, he or she must first develop an intuitive feeling. The most effective way to do that is to follow a sufficient number of examples.

This book is written for students who want to learn but need a firm handholding. I hope they will find the book readable and easy to learn from.

Learning, as always, has to be done by the student herself or himself. No one can acquire mathematical skill without doing problems, the more the better. However, realistically students have a finite amount of time. They will be overwhelmed if problems are too numerous, and frustrated if problems are too difficult. A common practice in textbooks is to list a large number of problems and let the instructor to choose a few for assignments. It seems to me that is not a confidence building strategy. A self-learning person would not know what to choose. Therefore a moderate number of not overly difficult problems, with answers, are selected at the end of each chapter. Hopefully after the student has successfully solved all of them, he or she will be encouraged to seek more challenging ones. There are plenty of problems in other books. Of course, an instructor can always assign more problems at levels suitable to the class.

On certain topics, I went farther than most other similar books, not in the sense of esoteric sophistication, but in making sure that the student can carry out the actual calculation. For example, the diagonalization of a degenerate hermitian matrix is of considerable importance in many fields. Yet to make it clear in a succinct way is not easy. I used several pages to give a detailed explanation of a specific example.

Professor I.I. Rabi used to say "All textbooks are written with the principle of least astonishment." Well, there is a good reason for that. After all, textbooks are supposed to explain away the mysteries and make the profound obvious. This book is no exception. Nevertheless, I still hope the reader will find something in this book exciting.

This volume consists of three chapters on complex analysis and three chapters on theory of matrices. In subsequent volumes, we will discuss vector and tensor analysis, ordinary differential equations and Laplace transforms, Fourier analysis and partial differential equations. Students are supposed to have already completed two or three semesters of calculus and a year of college physics.

This book is dedicated to my students. I want to thank my A and B students, their diligence and enthusiasm have made teaching enjoyable and worthwhile. I want to thank my C and D students, their difficulties and mistakes made me search for better explanations.

I want to thank Brad Oraw for drawing many figures in this book, and Mathew Hacker for helping me to typeset the manuscript.

I want to express my deepest gratitude to Professor S.H. Patil, Indian Institute of Technology, Bombay. He has read the entire manuscript and provided many excellent suggestions. He has also checked the equations and the problems and corrected numerous errors. Without his help and encouragement, I doubt this book would have been.

The responsibility for remaining errors is, of course, entirely mine. I will greatly appreciate if they are brought to my attention.

Tacoma, Washington *K.T. Tang*
October 2005

Contents

Part I Complex Analysis

Part I

Complex Analysis

1

Complex Numbers

The most compact equation in all of mathematics is surely

$$e^{i\pi} + 1 = 0. \tag{1.1}$$

In this equation, the five fundamental constants coming from four major branches of classical mathematics – arithmetic $(0, 1)$, algebra (i), geometry (π), and analysis (e), – are connected by the three most important mathematic operations – addition, multiplication, and exponentiation – into two nonvanishing terms.

The reader is probably aware that (1.1) is but one of the consequences of the miraculous Euler formula (discovered around 1740 by Leonhard Euler)

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.2}$$

When $\theta = \pi,\ \cos\pi = -1,$ and $\sin\pi = 0$, it follows that $e^{i\pi} = -1$.

Much of the computations involving complex numbers are based on the Euler formula. To provide a proper setting for the discussion of this formula, we will first present a sketch of our number system and some historic background. This will also give us a framework to review some of the basic mathematical operations.

1.1 Our Number System

Any one who encounters for the first time these equations cannot help but be intrigued by the strange properties of the numbers such as e and i. But strange is relative, with sufficient familiarity, the strange object of yesterday becomes the common thing of today. For example, nowadays no one will be bothered by the negative numbers, but for a long time negative numbers were regarded as "strange" or "absurd." For 2000 years, mathematics thrived without negative. The Greeks did not recognize negative numbers and did not need them. Their main interest was geometry, for the description of which positive numbers are entirely sufficient. Even after Hindu mathematician Brahmagupta "invented" zero around 628, and negative numbers were interpreted as a loss instead of a gain in financial matters, medieval Europe mostly ignored them.

Indeed, so long as one regards subtraction as an act of "taken away," negative numbers are absurd. One cannot take away, say, three apples from two.

Only after the development of the axiomatic algebra, the full acceptance of negative numbers into our number system was made possible. It is also within the framework of axiomatic algebra, irrational numbers and complex numbers are seen to be natural parts of our number system.

By axiomatic method, we mean the step by step development of a subject from a small set of definitions and a chain of logical consequences derived from them. This method had long been followed in geometry, ever since the Greeks established it as a rigorous mathematical discipline.

1.1.1 Addition and Multiplication of Integers

We start with the assumption that we know what integers are, what zero is, and how to count. Although mathematicians could go even further back and describe the theory of sets in order to derive the properties of integers, we are not going in that direction.

We put the integers on a line with increasing order as in the following diagram:

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\cdots \\
 & & \downarrow & & & \uparrow & & \\
 & & 2 & - & - & - & &
\end{array}
$$

If we start with certain integer a, and we count successively one unit b times to the right, the number we arrive at we call $a+b$, and that defines *addition* of integers. For example, starting at 2, and going up 3 units, we arrive at 5. So 5 is equal to $2+3$.

Once we have defined addition, then we can consider this: if we start with nothing and add a to it, b times in succession, we call the result *multiplication* of integers; we call it b *times* a.

Now as a consequence of these definitions it can be easily shown that these operations satisfy certain simple rules concerning the order in which the computations can proceed. They are the familiar commutative, associative, and distributive laws

$$
\begin{array}{cc}
a+b=b+a & \text{Commutative Law of Addition} \\
a+(b+c)=(a+b)+c & \text{Associative Law of Addition} \\
ab=ba & \text{Commutative Law of Multiplication} \\
(ab)c=a(bc) & \text{Associative Law of Multiplication} \\
a(b+c)=ab+ac & \text{Distributive Law.}
\end{array}
\tag{1.3}
$$

These rules characterize the *elementary algebra.* We say elementary algebra because there is a branch of mathematics called modern algebra in which some of the rules such as $ab = ba$ are abandoned, but we shall not discuss that.

Among the integers, 1 and 0 have special properties:

$$a + 0 = a,$$
$$a \cdot 1 = a.$$

So 0 is the additive identity and 1 is the multiplicative identity. Furthermore

$$0 \cdot a = 0$$

and if $ab = 0$, either a or/and b is zero.

Now we can also have a succession of multiplications: if we start with 1 and multiply by a, b times in succession, we call that *raising to power*: a^b. It follows from this definition that

$$(ab)^c = a^c b^c,$$
$$a^b a^c = a^{(b+c)},$$
$$(a^b)^c = a^{(bc)}.$$

These results are well known and we shall not belabor them.

1.1.2 Inverse Operations

In addition to the direct operation of addition, multiplication, and raising to a power, we have also the inverse operations, which are defined as follows. Let us assume a and c are given, and that we wish to find what values of b satisfy such equations as $a + b = c$, $ab = c$, $b^a = c$.

If $a+b = c$, b is defined as $c-a$, which is called *subtraction.* The operation called division is also clear: if $ab = c$, then $b = c/a$ defines division – a solution of the equation $ab = c$ "backwards."

Now if we have a power $b^a = c$ and we ask ourselves, "What is b?," it is called ath root of c : $b = \sqrt[a]{c}$. For instance, if we ask ourselves the following question, "What integer, raised to third power, equals 8?," then the answer is cube root of 8; it is 2. The direct and inverse operations are summarized as follows:

Operation		Inverse Operation	
(a) addition :	$a + b = c$	(a') subtraction :	$b = c - a$
(b) multiplication :	$ab = c$	(b') division :	$b = c/a$
(c) power :	$b^a = c$	(c') root :	$b = \sqrt[a]{c}$

Insoluble Problems

When we try to solve simple algebraic equations using these definitions, we soon discover some insoluble problems, such as the following. Suppose we try to solve the equation $b = 3 - 5$. That means, according to our definition of subtraction, that we must find a number which, when added to 5, gives 3. And of course there is no such number, because we consider only positive integers; this is an insoluble problem.

1.1.3 Negative Numbers

In the grand design of algebra, the way to overcome this difficulty is to broaden the number system through *abstraction* and *generalization*. We abstract the original definitions of addition and multiplication from the rules and integers. We assume the rules to be true in general on a wider class of numbers, even though they are originally derived on a smaller class. Thus, rather using the integers to symbolically define the rules, we use the rules as the definition of the symbols, which then represent a more general kind of number. As an example, by working with the rules alone we can show that $3 - 5 = 0 - 2$. In fact we can show that one can make all subtractions, provided we define a whole set of new numbers: $0-1,\ 0-2,\ 0-3,\ 0-4$, and so on (abbreviated as $-1,\ -2,\ -3,\ -4, \ldots$), called the negative numbers.

So we have increased the range of objects over which the rules work, but the meaning of the symbols is different. One cannot say, for instance, that -2 times 5 really means to add 5 together successively -2 times. That means nothing. But we require the negative numbers to obey all the rules.

For example, we can use the rules to show that -3 times -5 is equal to 15. Let $x = -3(-5)$, this is equivalent to $x + 3(-5) = 0$, or $x + 3\,(0 - 5) = 0$. By the rules, we can write this equation as

$$x + 0 - 15 = (x + 0) - 15 = x - 15 = 0.$$

Thus, $x = 15$. Therefore negative a times negative b is equal to positive ab,

$$(-a)(-b) = ab.$$

An interesting problem comes up in taking powers. Suppose we wish to discover what $a^{(3-5)}$ means. We know that $3-5$ is a solution of the problem, $(3 - 5) + 5 = 3$. Therefore

$$a^{(3-5)+5} = a^3.$$

Since

$$a^{(3-5)+5} = a^{(3-5)}a^5 = a^3$$

it follows that:

$$a^{(3-5)} = a^3/a^5.$$

Thus, in general

$$a^{n-m} = \frac{a^n}{a^m}.$$

If $n = m$, we have

$$a^0 = 1.$$

In addition, we found out what it means to raise a negative power. Since

$$3 - 5 = -2, \quad a^3/a^5 = \frac{1}{a^2}.$$

So

$$a^{-2} = \frac{1}{a^2}.$$

If our number system consists of only positive and negative integers, then $1/a^2$ is a meaningless symbol, because if a is a positive or negative integer, the square of it is greater than 1, and we do not know what we mean by 1 divided by a number greater than 1! So this is another insoluble problem.

1.1.4 Fractional Numbers

The great plan is to continue the process of generalization; whenever we find another problem that we cannot solve we extend our realm of numbers. Consider division: we cannot find a number which is an integer, even a negative integer, which is equal to the result of dividing 3 by 5. So we simply say that $3/5$ is another number, called fraction number. With the fraction number defined as a/b where a and b are integers and $b \neq 0$, we can talk about multiplying and adding fractions. For example, if $A = a/b$ and $B = c/b$, then by definition $bA = a$, $bB = c$, so $b(A + B) = a + c$.Thus, $A + B = (a + c)/b$. Therefore

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}.$$

Similarly, we can show

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}.$$

It can also be readily shown that fractional numbers satisfy the rules defined in (1.3). For example, to prove the commutative law of multiplication, we can start with

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{c}{d} \times \frac{a}{b} = \frac{ca}{db}.$$

Since a, b, c, d are integers, so $ac = ca$ and $bd = db$. Therefore $\frac{ac}{bd} = \frac{ca}{db}$. It follows that:

$$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}.$$

Take another example of powers: What is $a^{3/5}$? We know only that $(3/5)5 = 3$, since that was the definition of $3/5$. So we know also that

$$(a^{(3/5)})^5 = a^{(3/5)(5)} = a^3.$$

Then by the definition of roots we find that

$$a^{(3/5)} = \sqrt[5]{a^3}.$$

In this way we can define what we mean by putting fractions in the various symbols. It is a remarkable fact that all the rules still work for positive and negative integers, as well as for fractions!

Historically, the positive integers and their ratios (the fractions) were embraced by the ancients as natural numbers. These natural numbers together with their negative counter parts are known as *rational numbers* in our present day language.

The Greeks, under the influence of the teaching of Pythagoras, elevated fractional numbers to the central pillar of their mathematical and philosophical system. They believed that fractional numbers are prime cause behind everything in the world, from the laws of musical harmony to the motion of planets. So it was quite a shock when they found that there are numbers that cannot be expressed as a fraction.

1.1.5 Irrational Numbers

The first evidence of the existence of the irrational number (a number that is not a rational number) came from finding the length of the diagonal of a unit square. If the length of the diagonal is x, then by Pythagorean theorem $x^2 = 1^2 + 1^2 = 2$. Therefore $x = \sqrt{2}$. When people assumed this number is equal to some fraction, say m/n where m and n have no common factors, they found this assumption leads to a contradiction.

The argument goes as follows. If $\sqrt{2} = m/n$, then $2 = m^2/n^2$, or $2n^2 = m^2$. This means m^2 is an even integer. Furthermore, m itself must also be an even integer, since the square of an odd number is always odd. Thus $m = 2k$ for some integer k. It follows that $2n^2 = (2k)^2$, or $n^2 = 2k^2$. But this means n is also an even integer. Therefore, m and n have a common factor of 2, contrary to the assumption that they have no common factors. Thus $\sqrt{2}$ cannot be a fraction.

This was shocking to the Greeks, not only because of philosophical arguments, but also because mathematically, fractions form a dense set of numbers. By this we mean that between any two fractions, no matter how close, we can always squeeze in another. For example

$$\frac{1}{100} = \frac{2}{200} > \frac{2}{201} > \frac{2}{202} = \frac{1}{101}.$$

So we find $\frac{2}{201}$ between $\frac{1}{100}$ and $\frac{1}{101}$. Now between $\frac{1}{100}$ and $\frac{2}{201}$, we can squeeze in $\frac{4}{401}$, since

$$\frac{1}{100} = \frac{4}{400} > \frac{4}{401} > \frac{4}{402} = \frac{2}{201}.$$

This process can go on ad infinitum. So it seems only natural to conclude – as the Greeks did – that fractional numbers are continuously distributed on the number line. However, the discovery of irrational numbers showed that fractions, despite of their density, leave "holes" along the number line.

To bring the irrational numbers into our number system is in fact quite the most difficult step in the processes of generalization. A fully satisfactory theory of irrational numbers was not given until 1872 by Richard Dedekind (1831–1916), who made a careful analysis of continuity and ordering. To make the set of real numbers a continuum, we need the irrational numbers to fill the "holes" left by the rational numbers on the number line. A real number is any number that can be written as a decimal. There are three types of decimals: terminating, nonterminating but repeating, and nonterminating and nonrepeating. The first two types represent rational numbers, such as $\frac{1}{4} = 0.25$; $\frac{2}{3} = 0.666\ldots$. The third type represents irrational numbers, like $\sqrt{2} = 1.4142135\ldots$.

From a practical point of view, we can always approximate an irrational number by truncating the unending decimal. If higher accuracy is needed, we simply take more decimal places. Since any decimal when stopped somewhere is rational, this means that an irrational number can be represented by a sequence of rational numbers with progressively increasing accuracy. This is good enough for us to perform mathematical operations with irrational numbers.

1.1.6 Imaginary Numbers

We go on in the process of generalization. Are there any other insoluble equations? Yes, there are. For example, it is impossible to solve this equation: $x^2 = -1$. The square of no rational, of no irrational, of nothing that we have discovered so far, is equal to -1. So again we have to generalize our numbers to still a wider class.

This time we extend our number system to include the solution of this equation, and introduce the symbol i for $\sqrt{-1}$ (engineers call it j to avoid confusion with current). Of course some one could call it $-\mathrm{i}$ since it is just as good a solution. The only property of i is that $\mathrm{i}^2 = -1$. Certainly, $x = -\mathrm{i}$ also satisfies the equation $x^2 + 1 = 0$. Therefore it must be true that any equation we can write is equally valid if the sign of i is changed everywhere. This is called taking the *complex conjugate*.

We can make up numbers by adding successively i's, and multiplying i's by numbers, and adding other numbers and so on, according to all our rules. In this way we find that numbers can all be written as $a + \mathrm{i}b$, where a and b are real numbers, i.e., the numbers we have defined up until now. The number i is called the *unit imaginary number*. Any real multiple of i is called *pure imaginary*. The most general number is of course of the form $a + \mathrm{i}b$ and is called a *complex number*. Things do not get any worse if we add and multiply two such numbers. For example

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}. \tag{1.4}$$

In accordance with the distributive law, the multiplication of two complex number is defined as

$$\begin{aligned} (a + b\mathrm{i})\,(c + d\mathrm{i}) &= ac + a(d\mathrm{i}) + (b\mathrm{i})c + (b\mathrm{i})(d\mathrm{i}) \\ &= ac + (ad)\mathrm{i} + (bc)\mathrm{i} + (bd)\mathrm{ii} = (ac - bd) + (ad + bc)\mathrm{i}, \end{aligned} \tag{1.5}$$

since $\mathrm{ii} = \mathrm{i}^2 = -1$. Therefore all the numbers have this mathematical form.

It is customary to use a single letter, z, to denote a complex number $z = a + b\mathrm{i}$. Its real and imaginary parts are written as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. With this notation, $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$. The equation $z_1 = z_2$ holds if and only if

$$\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2) \quad \text{and} \quad \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\,.$$

Thus any equation involving complex numbers can be interpreted as a pair of real equations.

The complex conjugate of the number $z = a + b\mathrm{i}$ is usually denoted as either z^*, or $\overline{z}$, and is given by $z^* = a - b\mathrm{i}$. An important relation is that the product of a complex number and its complex conjugate is a real number

$$zz^* = (a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + b^2.$$

With this relation, the division of two complex numbers can also be written as the sum of a real part and an imaginary part

$$\frac{a + b\mathrm{i}}{c + d\mathrm{i}} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}}\frac{c - d\mathrm{i}}{c - d\mathrm{i}} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\mathrm{i}.$$

Example 1.1.1. Express the following in the form of $a + b\mathrm{i}$:

(a) $(6 + 2\mathrm{i}) - (1 + 3\mathrm{i})$, $\quad$ (b) $(2 - 3\mathrm{i})(1 + \mathrm{i})$,

(c) $\left(\dfrac{1}{2 - 3\mathrm{i}}\right)\left(\dfrac{1}{1 + \mathrm{i}}\right)$.

Solution 1.1.1.

$$(a)\;\; (6+2\mathrm{i})-(1+3\mathrm{i}) = (6-1)+\mathrm{i}(2-3) = 5-\mathrm{i}.$$

$$\begin{aligned}(b)\;\; (2-3\mathrm{i})(1+\mathrm{i}) &= 2\,(1+\mathrm{i})-3\mathrm{i}(1+\mathrm{i}) = 2+2\mathrm{i}-3\mathrm{i}-3\mathrm{i}^2 \\ &= (2+3)+\mathrm{i}(2-3) = 5-\mathrm{i}.\end{aligned}$$

$$\begin{aligned}(c)\;\; \left(\frac{1}{2-3\mathrm{i}}\right)\left(\frac{1}{1+\mathrm{i}}\right) &= \frac{1}{(2-3\mathrm{i})\,(1+\mathrm{i})} = \frac{1}{5-\mathrm{i}} \\ &= \frac{5+\mathrm{i}}{(5-\mathrm{i})\,(5+\mathrm{i})} = \frac{5+\mathrm{i}}{5^2-\mathrm{i}^2} = \frac{5}{26}+\frac{1}{26}\mathrm{i}.\end{aligned}$$

Historically, Italian mathematician Girolamo Cardano was credited as the first to consider the square root of a negative number in 1545 in connection with solving quadratic equations. But after introducing the imaginary numbers, he immediately dismissed them as "useless." He had a good reason to think that way. At Cardano's time, mathematics was still synonymous with geometry. Thus the quadratic equation $x^2 = mx + c$ was thought as a vehicle to find the intersection points of the parabola $y = x^2$ and the line $y = mx + c$. For an equation such as $x^2 = -1$, the horizontal line $y = -1$ will obviously not intersect the parabola $y = x^2$ which is always positive. The absence of the intersection was thought as the reason of the occurrence of the imaginary numbers.

It was the cubic equation that forced complex numbers to be taken seriously. For a cubic curve $y = x^3$, the values of y go from $-\infty$ to $+\infty$. A line will always hit the curve at least once. In 1572, Rafael Bombeli considered the equation

$$x^3 = 15x + 4,$$

which clearly has a solution of $x = 4$. Yet at the time, it was known that this kind of equation could be solved by the following formal procedure. Let $x = a + b$, then

$$x^3 = (a+b)^3 = a^3 + 3ab(a+b) + b^3,$$

which can be written as

$$x^3 = 3abx + (a^3 + b^3).$$

The problem will be solved, if we can find a set of values a and b satisfying the conditions

$$3ab = 15 \quad \text{and} \quad a^3 + b^3 = 4.$$

Since $a^3b^3 = 5^3$ and $b^3 = 4 - a^3$, we have

$$a^3(4 - a^3) = 5^3,$$

which is a quadratic equation in a^3

$$\left(a^3\right)^2 - 4a^3 + 125 = 0.$$

The solution of such an equation was known for thousands of years,

$$a^3 = \frac{1}{2}\left(4 \pm \sqrt{16 - 500}\right) = 2 \pm 11\mathrm{i}.$$

It follows that:

$$b^3 = 4 - a^3 = 2 \mp 11\mathrm{i}.$$

Therefore

$$x = a + b = (2 + 11\mathrm{i})^{1/3} + (2 - 11\mathrm{i})^{1/3}.$$

Clearly, the interpretation that the appearance of imaginary numbers signifies no solution of the geometric problem is not valid. In order to have the solution come out to equal 4, Bombeli assumed

$$(2 + 11\mathrm{i})^{1/3} = 2 + b\mathrm{i}; \qquad (2 - 11\mathrm{i})^{1/3} = 2 - b\mathrm{i}.$$

To justify this assumption, he had to use the rules of addition and multiplication of complex numbers. With the rules listed in (1.4) and (1.5), it can be readily shown that

$$\begin{aligned}(2 + b\mathrm{i})^3 &= 8 + 3\,(4)\,(b\mathrm{i}) + 3(2)\,(b\mathrm{i})^2 + (b\mathrm{i})^3 \\ &= \left(8 - 6b^2\right) + \left(12b - b^3\right)\mathrm{i}.\end{aligned}$$

With $b = \pm 1$, he obtained

$$(2 \pm \mathrm{i})^3 = 2 \pm 11\mathrm{i},$$

and

$$x = (2 + 11\mathrm{i})^{1/3} + (2 - 11\mathrm{i})^{1/3} = 2 + \mathrm{i} + 2 - \mathrm{i} = 4.$$

Thus he established that problems with real coefficients required complex arithmetic for solutions.

Despite Bombelli's work, complex numbers were greeted with suspicion, even hostility for almost 250 years. Not until the beginning of the 19th century, complex numbers were fully embraced as members of our number system. The acceptance of complex numbers was largely due to the work and reputation of Gauss.

Karl Friedrich Gauss (1777–1855) of Germany was given the title of "the prince of mathematics" by his contemporaries as a tribute to his great achievements in almost every branch of mathematics. At the age of 22, Gauss in his doctoral dissertation gave the first rigorous proof of what we now call the Fundamental Theorem of Algebra. It says that a polynomial of degree n always has exactly n complex roots. This shows that complex numbers are not only necessary to solve a general algebraic equation, they are also sufficient. In other words, with the invention of i, every algebraic equation can be solved. This is a fantastic fact. It is certainly not self-evident. In fact, the process by which our number system is developed would make us think that we will have to keep on inventing new numbers to solve yet unsolvable equations. It is a miracle that this is not the case. With the last invention of i, our number system is complete. Therefore a number, no matter how complicated it looks, can always be reduced to the form of $a + b\mathrm{i}$, where a and b are real numbers.

1.2 Logarithm

1.2.1 Napier's Idea of Logarithm

Rarely a new idea was embraced so quickly by the entire scientific community with such enthusiasm as the invention of logarithm. Although it was merely a device to simplify computation, its impact on scientific developments could not be overstated.

Before 17th century scientists had to spend much of their time doing numerical calculations. The Scottish baron, John Napier (1550–1617) thought to relieve this burden as he wrote: "Seeing there is nothing that is so troublesome to mathematical practice than multiplications, divisions, square and cubical extractions of great numbers,......I began therefore in my mind by what certain and ready art I might remove those hinderance." His idea was this: if we could write any number as a power of some given, fixed number b (later to be called base), then multiplication of numbers would be equivalent to addition of their exponents. He called the power logarithm.

In modern notation, this works as follows. If

$$b^{x_1} = N_1; \qquad b^{x_2} = N_2$$

then by definition

$$x_1 = \log_b N_1; \quad x_2 = \log_b N_2.$$

Obviously

$$x_1 + x_2 = \log_b N_1 + \log_b N_2.$$

Since

$$b^{x_1+x_2} = b^{x_1} b^{x_2} = N_1 N_2$$

again by definition

$$x_1 + x_2 = \log_b N_1 N_2.$$

Therefore

$$\log_b N_1 N_2 = \log_b N_1 + \log_b N_2.$$

Suppose we have a table, in which N and $\log_b N$ (the power x) are listed side by side. To multiply two numbers N_1 and N_2, you first look up $\log_b N_1$ and $\log_b N_2$ in the table. You then add the two numbers. Next, find the number in the body of the table that matches the sum, and read backward to get the product $N_1 N_2$.

Similarly, we can show

$$\log_b \frac{N_1}{N_2} = \log_b N_1 - \log_b N_2,$$

$$\log_b N^n = n \log_b N, \qquad \log_b N^{1/n} = \frac{\log_b N}{n}.$$

Thus, division of numbers would be equivalent to subtraction of their exponents, raising a number to nth power would be equivalent to multiplying the exponent by n, and finding the nth root of a number would be equivalent to dividing the exponent by n. In this way the drudgery of computations is greatly reduced.

Now the question is, with what base b should we compute. Actually it makes no difference what base is used, as long as it is not exactly equal to 1. We can use the same principle all the time. Besides, if we are using logarithms to any particular base, we can find logarithms to any other base merely by multiplying a factor, equivalent to a change of scale. For example, if we know the logarithm of all numbers with base b, we can find the logarithm of N with base a. First if $a = b^x$, then by definition, $x = \log_b a$, therefore

$$a = b^{\log_b a}. \tag{1.6}$$

To find $\log_a N$, first let $y = \log_a N$. By definition $a^y = N$. With a given by (1.6), we have

$$\left(b^{\log_b a}\right)^y = b^{y \log_b a} = N.$$

Again by definition (or take logarithm of both sides of the equation)

$$y \log_b a = \log_b N.$$

Thus

$$y = \frac{1}{\log_b a} \log_b N.$$

Since $y = \log_a N$, it follows:

$$\log_a N = \frac{1}{\log_b a} \log_b N.$$

This is known as change of base. Having a table of logarithm with base b will enable us to calculate the logarithm to any other base.

In any case, the key is, of course, to have a table. Napier chose a number slightly less than one as the base and spent 20 years to calculate the table. He published his table in 1614. His invention was quickly adopted by scientists all across Europe and even in far away China. Among them was the astronomer Johannes Kepler, who used the table with great success in his calculations of the planetary orbits. These calculations became the foundation of Newton's classical dynamics and his law of gravitation.

1.2.2 Briggs' Common Logarithm

Henry Briggs (1561–1631), a professor of geometry in London, was so impressed by Napier's table, he went to Scotland to meet the great inventor in person. Briggs suggested that a table of base 10 would be more convenient. Napier readily agreed. Briggs undertook the task of additional computations. He published his table in 1624. For 350 years, the logarithmic table and the slide rule (constructed with the principle of logarithm) were indispensable tools of every scientist and engineer.

The logarithm in Briggs' table is now known as the common logarithm. In modern notation, if we write $x = \log N$ without specifying the base, it is understood that the base is 10, and $10^x = N$.

Today logarithmic tables are replaced by hand-held calculators, but logarithmic function remains central to mathematical sciences.

It is interesting to see how logarithms were first calculated. In addition to historic interests, it will help us to gain some insights into our number system.

Since a simple process for taking square roots was known, Briggs computed successive square roots of 10. A sample of the results is shown in Table 1.1. The powers (x) of 10 are given in the first column and the results, 10^x, are given in the second column. For example, the second row is the square root of 10, that is $10^{1/2} = \sqrt{10} = 3.16228$. The third row is the square root of the square root of 10, $\left(10^{1/2}\right)^{1/2} = 10^{1/4} = 1.77828$. So on and so forth, we get a series of successive square roots of 10. With a hand-held calculator, you can readily verify these results.

In the table we noticed that when 10 is raised to a very small power, we get 1 plus a small number. Furthermore, the small numbers that are added

Table 1.1. Successive square roots of ten

x ($\log N$)	10^x (N)	$(10^x - 1)/x$
1	10.0	9.00
$\frac{1}{2} = 0.5$	3.16228	4.32
$(\frac{1}{2})^2 = 0.25$	1.77828	3.113
$(\frac{1}{2})^3 = 0.125$	1.33352	2.668
$(\frac{1}{2})^4 = 0.0625$	1.15478	2.476
$(\frac{1}{2})^5 = 0.03125$	1.074607	2.3874
$(\frac{1}{2})^6 = 0.015625$	1.036633	2.3445
$(\frac{1}{2})^7 = 0.0078125$	1.018152	2.3234
$(\frac{1}{2})^8 = 0.00390625$	1.0090350	2.3130
$(\frac{1}{2})^9 = 0.001953125$	1.0045073	2.3077
$(\frac{1}{2})^{10} = 0.00097656$	1.0022511	2.3051
$(\frac{1}{2})^{11} = 0.00048828$	1.0011249	2.3038
$(\frac{1}{2})^{12} = 0.00024414$	1.0005623	2.3032
$(\frac{1}{2})^{13} = 0.00012207$	1.000281117	2.3029
$(\frac{1}{2})^{14} = 0.000061035$	1.000140548	2.3027
$(\frac{1}{2})^{15} = 0.0000305175$	1.000070272	2.3027
$(\frac{1}{2})^{16} = 0.0000152587$	1.000035135	2.3026
$(\frac{1}{2})^{17} = 0.0000076294$	1.0000175675	2.3026

to 1 begins to look as though we are merely dividing by 2 each time we take a square root. In other words, it looks that when x is very small, $10^x - 1$ is proportional to x. To find the proportionality constant, we list $(10^x - 1)/x$ in column 3. At the top of the table, these ratios are not equal, but as they come down, they get closer and closer to a constant value. To the accuracy of five significant digits, the proportional constant is equal to 2.3026. So we find that when s is very small

$$10^s = 1 + 2.3026s. \tag{1.7}$$

Briggs computed successively 27 square roots of 10, and used (1.7) to obtain another 27 squares roots.

Since $10^x = N$ means $x = \log N$, the first column in Table 1.1 is also the logarithm of the corresponding number in the second column. For example, the second row is the square root of 10, that is $10^{1/2} = 3.16228$. Then by definition, we know

$$\log(3.16228) = 0.5.$$

If we want to know the logarithm of a particular number N, and N is not exactly the same as one of the entries in the second column, we have to break up N as a product of a series of numbers which are entries of the table. For example, suppose we want to know the logarithm of 1.2. Here is what we do. Let $N = 1.2$, and we are going to find a series of n_i in column 2 such that

$$N = n_1 n_2 n_3 \cdots .$$

Since all n_i are greater than one, so $n_i < N$. The number in column 2 closest to 1.2 satisfying this condition is 1.15478, So we choose $n_1 = 1.15478$, and we have

$$\frac{N}{n_1} = \frac{1.2}{1.15478} = 1.039159 = n_2 n_3 \cdots .$$

The number smaller than and closest to 1.039159 is 1.036633. So we choose $n_2 = 1.036633$, thus

$$\frac{N}{n_1 n_2} = \frac{1.039159}{1.036633} = 1.0024367.$$

With $n_3 = 1.0022511$, we have

$$\frac{N}{n_1 n_2 n_3} = \frac{1.0024367}{1.0022511} = 1.0001852.$$

The plan is to continue this way until the right-hand side is equal to one. But most likely, sooner or later, the right-hand side will fall beyond the table and is still not exactly equal to one. In our particular case, we can go down a couple of more steps. But for the purpose of illustration, let us stop here. So

$$N = n_1 n_2 n_3 (1 + \Delta n),$$

where $\Delta n = 0.0001852$. Now

$$\log N = \log n_1 + \log n_2 + \log n_3 + \log(1 + \Delta n).$$

The terms on the right-hand side, except the last one, can be read from the table. For the last term, we will make use of (1.7). By definition, if s is very small, (1.7) can be written as

$$s = \log(1 + 2.3026s).$$

Let $\Delta n = 2.3026s$, so $s = \frac{\Delta n}{2.3026} = \frac{0.0001852}{2.3026} = 8.04 \times 10^{-5}$. It follows:

$$\log(1 + \Delta n) = \log[1 + 2.3026 \left(8.04 \times 10^{-5}\right)] = 8.04 \times 10^{-5}.$$

With $\log n_1 = 0.0625$, $\log n_2 = 0.015625$, $\log n_3 = 0.0009765$ from the table, we arrived at

$$\log(1.2) = 0.0625 + 0.015625 + 0.0009765 + 0.0000804 = 0.0791819.$$

The value of log(1.2) should be 0.0791812. Clearly if we have a larger table we can have as many accurate digits as we want. In this way Briggs calculated the logarithms to 16 decimal places and reduced them to 14 when he published his table, so there were no rounding errors. With minor revisions, Briggs' table remained the basis for all subsequent logarithmic tables for the next 300 years.

1.3 A Peculiar Number Called e

1.3.1 The Unique Property of e

Equation (1.7) expresses a very interesting property of our number system. If we let $t = 2.3026s$, then for a very small t, (1.7) becomes

$$10^{\frac{t}{2.3026}} = 1 + t. \tag{1.8}$$

To simplify the writing, let us denote

$$10^{\frac{1}{2.3026}} = \mathrm{e}. \tag{1.9}$$

Thus (1.8) says that e raised to a very small power is equal to one plus the small power

$$\mathrm{e}^t = 1 + t \quad \text{for } t \to 0. \tag{1.10}$$

Because of this, we find the derivative of e^x is equal to itself.

Recall the definition of the derivative of a function:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

So

$$\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\mathrm{e}^{x+\Delta x} - \mathrm{e}^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\mathrm{e}^x(\mathrm{e}^{\Delta x} - 1)}{\Delta x}.$$

Now Δx approaches zero as a limit, certainly it is very small, so we can write (1.10) as

$$\mathrm{e}^{\Delta x} = 1 + \Delta x.$$

Thus

$$\frac{\mathrm{e}^x(\mathrm{e}^{\Delta x} - 1)}{\Delta x} = \frac{\mathrm{e}^x(1 + \Delta x - 1)}{\Delta x} = \mathrm{e}^x.$$

Therefore

$$\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} = \mathrm{e}^x. \tag{1.11}$$

The function e^x (or written as $\exp(x)$) is generally called the natural exponential function, or simply the exponential function. Not only is the exponential function equal to its own derivative, it is the only function (apart from a multiplication constant) that has this property. Because of this, the exponential function plays a central role in mathematics and sciences.

1.3.2 The Natural Logarithm

If $\mathrm{e}^y = x$, then by definition

$$y = \log_{\mathrm{e}} x.$$

The logarithm to the base e is known as the natural logarithm. It appears with amazing frequency in mathematics and its applications. So we give it a special symbol. It is written as $\ln x$. That is

$$y = \log_{\mathrm{e}} x = \ln x.$$

Thus

$$\mathrm{e}^{\ln x} = x.$$

Furthermore,

$$\ln \mathrm{e}^x = x \ln \mathrm{e} = x.$$

In this sense, the exponential function and the natural logarithm are inverses of each other.

Example 1.3.1. Find the value of $\ln 10$.

Solution 1.3.1. Since

$$10^{\frac{1}{2.3026}} = \mathrm{e}, \quad \Rightarrow \quad 10 = \mathrm{e}^{2.3026}$$

it follows:

$$\ln 10 = \ln \mathrm{e}^{2.3026} = 2.3026.$$

The derivative of $\ln x$ is of special interests.

$$\frac{\mathrm{d}(\ln x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x},$$

$$\ln(x + \Delta x) - \ln x = \ln \frac{x + \Delta x}{x} = \ln\left(1 + \frac{\Delta x}{x}\right).$$

Now (1.10) can be written as

$$t = \ln(1 + t)$$

for a very small t. Since Δx approaches zero as a limit, for any fixed x, $\frac{\Delta x}{x}$ can certainly be made as small as we wish. Therefore, we can set $\frac{\Delta x}{x} = t$, and conclude

$$\frac{\Delta x}{x} = \ln\left(1 + \frac{\Delta x}{x}\right).$$

Thus,

$$\frac{\mathrm{d}(\ln x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\frac{\Delta x}{x} = \frac{1}{x}.$$

This in turn means

$$\mathrm{d}\left(\ln x\right) = \frac{\mathrm{d}x}{x}$$

or

$$\int \frac{\mathrm{d}x}{x} = \ln x + c, \tag{1.12}$$

where c is the constant of integration. It is well known that because of

$$\frac{\mathrm{d}x^{n+1}}{\mathrm{d}x} = (n+1)x^n,$$

we have

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{(n+1)} + c.$$

This formula holds for all values of n except for $n = -1$, since then the denominator $n+1$ is zero. This had been a difficult problem, but now we see that (1.12) provides the "missing case."

In numerous phenomena, ranging from population growth to the decay of radioactive material, in which the rate of change of some quantity is proportional to the quantity itself. Such phenomenon is governed by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky,$$

where k is a constant that is positive if y is increasing and negative if y is decreasing. To solve this equation, we write it as

$$\frac{\mathrm{d}y}{y} = k\,\mathrm{d}t$$

and then integrate both sides to get

$$\ln y = kt + c, \quad \textit{or} \quad y = \mathrm{e}^{kt+c} = \mathrm{e}^{kt}\mathrm{e}^{c}.$$

If y_0 denotes the value of y when $t = 0$, then $y_0 = \mathrm{e}^c$ and

$$y = y_0\,\mathrm{e}^{kt}.$$

This equation is called the law of exponential change.

1.3.3 Approximate Value of e

The number e is found of such great importance, but what is the numerical value of e, which we have, so far, defined as $10^{(1/2.3025)}$? We can use our table of successive square root of 10 to calculate this number. The powers of 10 are given in the first column of Table 1.1. If we can find a series of numbers $n_1,\ n_2,\ n_3, \ldots$ in this column, such that

$$\frac{1}{2.3026} = n_1 + n_2 + n_3 + \cdots ,$$

then

$$10^{\frac{1}{2.3026}} = 10^{n_1+n_2+n_3+\cdots} = 10^{n_1}10^{n_2}10^{n_3}\cdots .$$

We can read from the second column of the table 10^{n_1}, and 10^{n_2}, and 10^{n_3} and so on, and multiply them together. Let us do just that.

$$\begin{aligned}\frac{1}{2.3026} = 0.43429 = 0.25 + 0.125 + 0.03125 + 0.015625 \\ + 0.0078125 + 0.00390625 + 0.00048828 + 0.00012207 \\ + 0.000061035 + 0.000026535.\end{aligned}$$

From the table we find $10^{0.25} = 1.77828,\ 10^{0.125} = 1.33352, \ldots$ etc. except for the last term for which we use (1.7). Thus

$$\begin{aligned}\mathrm{e} = 10^{\frac{1}{2.3026}} &= 1.77828 \times 1.33352 \times 1.074607 \times 1.036633 \times 1.018152 \\ &\times 1.009035 \times 1.0011249 \times 1.000281117 \times 1.000140548 \\ &\times (1 + 2.3026 \times 0.000026535) = 2.71826.\end{aligned}$$

Since $\frac{1}{2.3026}$ is only accurate to 5 significant digits, we cannot expect our result to be accurate more than that. (The accurate result is $2.71828\cdots$) Thus what we get is only an approximation. Is there a more precise definition of e? The answer is yes. We will discuss this question in the next section.

1.4 The Exponential Function as an Infinite Series

1.4.1 Compound Interest

The origins of the number e are not very clear. The existence of this peculiar number could be extracted from the logarithmic table as we did. In fact there is an indirect reference to e in the second edition of Napier's table. But most probably the peculiar property of the number e was noticed even earlier in connection with compound interest.

A sum of money invested at x percent annual interest rate (x expressed as a decimal, for example $x = 0.06$ for 6%) means that at the end of the year the sum grows by a factor $(1+x)$. Some banks compute the accrued interest not once a year but several times a year. For example, if an annual interest rate of x percent is compounded semiannually, the bank will use one-half of the annual rate as the rate per period. Hence, if P is the original sum, at the end of the half-year, the sum grows to $P\left(1+\frac{x}{2}\right)$, and at the end of the year the sum becomes

$$\left[P\left(1+\frac{x}{2}\right)\right]\left(1+\frac{x}{2}\right)=P\left(1+\frac{x}{2}\right)^2.$$

In the banking industry one finds all kinds of compounding schemes – annually, semiannually, quarterly, monthly, weekly, and even daily. Suppose the compounding is done n times a year, at the end of the year, the principal P will yield the amount

$$S=P\left(1+\frac{x}{n}\right)^n.$$

It is interesting to compare the amounts of money a given principal will yield after one year for different conversion periods. Table 1.2 shows that the amounts of money one will get for \$100 invested for 1 year at 6% annual interest rate at different conversion periods. The result is quite surprising. As we see, a principal of \$100 compounded daily or weekly yield practically the same. But will this pattern go on? Is it possible that no matter how large n is, the values of $(1+\frac{x}{n})^n$ will settle on the same number? To answer this question, we must use methods other than merely computing individual values. Fortunately, such a method is available. With the binomial formula,

$$\begin{aligned}(a+b)^n &= a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2\\ &\quad+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+\cdots+b^n,\end{aligned}$$

we have

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n &= 1+n\left(\frac{x}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2\\ &\quad+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3+\cdots+\left(\frac{x}{n}\right)^n\\ &= 1+x+\frac{(1-1/n)}{2!}x^2+\frac{(1-1/n)(1-2/n)}{3!}x^3+\cdots\left(\frac{x}{n}\right)^n.\end{aligned}$$

Now as $n\to\infty$, $\frac{k}{n}\to 0$. Therefore

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots. \tag{1.13}$$

becomes an infinite series. Standard tests for convergence show that this is a convergent series for all real values of x. In other words, the value of $(1+\frac{x}{n})^n$ does settle on a specific limit as n increase without bound.

Table 1.2. The yields of \$100 invested for 1 year at 6% annual interest rate at different conversion periods

	n	x/n	$100(1+x/n)^n$
Annually	1	0.06	106.00
Semiannually	2	0.03	106.09
Quarterly	4	0.015	106.136
Monthly	12	0.005	106.168
Weekly	52	0.0011538	106.180
Daily	365	0.0001644	106.183

1.4.2 The Limiting Process Representing e

In early 18th century, Euler used the letter e to represent the series (1.13) for the case of $x = 1$,

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots . \tag{1.14}$$

This choice, like many other symbols of his, such as i, π , $f(x)$, became universally accepted.

It is important to note that when we say that the limit of $\frac{1}{n}$ as $n\to\infty$ is 0 it does not mean that $\frac{1}{n}$ itself will ever be equal to 0, in fact, it will not. Thus, if we let $t=\frac{1}{n}$, then as $n\to\infty$, $t\to 0$. So (1.14) can be written as

$$\mathrm{e} = \lim_{t\to 0}(1+t)^{1/t} .$$

In words, it says that if t is very small, then

$$\mathrm{e}^t = \left[(1+t)^{1/t}\right]^t = 1+t, \qquad t\to 0.$$

This is exactly the same equation as shown in (1.10). Therefore, e is the same number previously written as $10^{1/2.3026}$. Now the formal definition of e is given by the limiting process

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n ,$$

which can be written as an infinite series as shown in (1.14). The series converges rather fast. With seven terms, it gives us 2.71825. This approximation can be improved by adding more terms until the desired accuracy is reached. Since it is monotonely convergent, each additional term brings it closer to the limit: $2.71828\cdots$.

1.4.3 The Exponential Function e^x

Raising e to x power, we have

$$\mathrm{e}^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}.$$

Let $nx = m$, then $\frac{x}{m} = \frac{1}{n}$. As n goes to ∞, so does m. Thus the above equation becomes

$$\mathrm{e}^x = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}.$$

Now m may not be an integer, but the binomial formula is equally valid for noninteger power (one of the early discoveries of Isaac Newton). Therefore by the same reason as in (1.13), we can express the exponential function as an infinite series,

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{1.15}$$

It is from this series that the numerical values of e^x are most easily obtained, the first few terms usually suffice to obtain the desired accuracy.

We have shown in (1.11) that the derivative of e^x must be equal to itself. This is clearly the case as we take derivative of (1.15) term by term,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \mathrm{e}^x.$$

1.5 Unification of Algebra and Geometry

1.5.1 The Remarkable Euler Formula

Leonhard Euler (1707–1783) was born in Basel, a border town between Switzerland, France, and Germany. He is one of the great mathematicians and certainly the most prolific scientist of all times. His immense output fills at least 70 volumes. In 1771, after he became blind, he published three volumes of a profound treatise of optics. For almost 40 years after his death, the Academy at St. Petersburg continued to publish his manuscripts. Euler played with formulas like a child playing toys, making all kinds of substitutions until he got something interesting. Often the results were sensational.

He took the infinite series of e^x, and boldly replaced the real variable x in (1.15) with the imaginary expression $\mathrm{i}\theta$ and got

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \cdots.$$

Since $\mathrm{i}^2 = -1$, $\mathrm{i}^3 = -\mathrm{i}$, $\mathrm{i}^4 = 1$, and so on, this equation became

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots .$$

He then changed the order of terms, collecting all the real terms separately from the imaginary terms, and arrived at the series

$$\mathrm{e}^{\mathrm{i}\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right).$$

Now it was already known in Euler's time that the two series appearing in the parentheses are the power series of the trigonometric functions $\cos\theta$ and $\sin\theta$, respectively. Thus Euler arrived at the remarkable formula (1.2)

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta,$$

which at once links the exponential function to ordinary trigonometry.

Strictly speaking, Euler played the infinite series rather loosely. Collecting all the real terms separately from the imaginary terms, he changed the order of terms. To do so with an infinite series can be dangerous. It may affect its sum, or even change a convergent series into a divergent series. But this result has withstood the test of rigor.

Euler derived hundreds of formulas, but this one is often called the most famous formula of all formulas. Feynman called it the amazing jewel.

1.5.2 The Complex Plane

The acceptance of complex number as a bona fide members of our number system was greatly helped by the realization that a complex number could be given a simple, concrete geometric interpretation. In a two-dimensional rectangular coordinate system, a point is specified by its x and y components. If we interpret the x and y axes as the real and imaginary axes, respectively, then the complex number $z = x + \mathrm{i}y$ is represented by the point (x, y). The horizontal position of the point is x, the vertical position of the point is y, as shown in Fig. 1.1. We can then add and subtract complex numbers by separately adding or subtracting the real and imaginary components. When thought in this way, the plane is called the complex plane, or the Argand plane.

This graphic representation was independently suggested around 1800 by Wessel of Norway, Argand of France, and Gauss. The publications by Wessel and by Argand went all but unnoticed, but the reputation of Gauss ensured wide dissemination and acceptance of the complex numbers as points in the complex plane.

At the time when this interpretation was suggested, the Euler formula (1.2) had already been known for at least 50 years. It might have played the

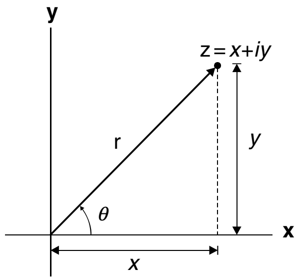


Fig. 1.1. Complex plane also known as Argand diagram. The real part of a complex number is along the x-axis, and the imaginary part, along the y-axis

role of guiding principle for this suggestion. The geometric interpretation of the complex number is certainly consistent with the Euler formula. We can derive the Euler formula by expressing $\mathrm{e}^{\mathrm{i}\theta}$ as a point in the complex plane.

Since the most general number is a complex number in the form of a real part plus an imaginary part, so let us express $\mathrm{e}^{\mathrm{i}\theta}$ as

$$\mathrm{e}^{\mathrm{i}\theta} = a(\theta) + \mathrm{i}b(\theta) \,. \tag{1.16}$$

Note that both the real part a and the imaginary part b must be functions of θ. Here θ is any real number. Changing i to $-$i, in both sides of this equation, we get the complex conjugate

$$\mathrm{e}^{-\mathrm{i}\theta} = a(\theta) - \mathrm{i}b(\theta) \,.$$

Since

$$\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\theta-\mathrm{i}\theta} = \mathrm{e}^{0} = 1,$$

it follows that

$$\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = (a+\mathrm{i}b)(a-\mathrm{i}b) = a^2 + b^2 = 1.$$

Furthermore

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}(a+\mathrm{i}b) = \mathrm{i}a - b$$

but

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathrm{i}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}a + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}b = a' + \mathrm{i}b',$$

equating the real part to real part and imaginary part to imaginary part of the last two equations, we have

$$a' = \frac{\mathrm{d}}{\mathrm{d}\theta}a = -b, \quad b' = \frac{\mathrm{d}}{\mathrm{d}\theta}b = a.$$

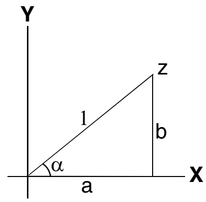


Fig. 1.2. The Argand diagram of the complex number $z = \mathrm{e}^{\mathrm{i}\theta} = a + \mathrm{i}b$. The distance between the origin and the point (a, b) must be 1

Thus

$$a'b = -b^2, \quad b'a = a^2$$

and

$$b'a - a'b = a^2 + b^2 = 1.$$

Now let $a(\theta)$ represent the abscissa (x-coordinate) and $b(\theta)$ represent the ordinate (y-coordinate) of a point in the complex plane as shown in Fig. 1.2. Let α be the angle between the x-axis and the vector from the origin to the point. Since the length of this vector is given by the Pythagorean theorem

$$r^2 = a^2 + b^2 = 1,$$

clearly

$$\cos\alpha = \frac{a(\theta)}{1} = a(\theta), \quad \sin\alpha = \frac{b(\theta)}{1} = b(\theta), \quad \tan\alpha = \frac{b(\theta)}{a(\theta)}. \tag{1.17}$$

Now

$$\frac{\mathrm{d}\tan\alpha}{\mathrm{d}\theta} = \frac{\mathrm{d}\tan\alpha}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = \frac{1}{\cos^2\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = \frac{1}{a^2}\frac{\mathrm{d}\alpha}{\mathrm{d}\theta},$$

but

$$\frac{\mathrm{d}\tan\alpha}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{b}{a}\right) = \frac{b'a - a'b}{a^2} = \frac{1}{a^2}.$$

It is clear from the last two equations that

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = 1.$$

In other words,

$$\alpha = \theta + c.$$

To determine the constant c, let us look at the case $\theta = 0$. Since $\mathrm{e}^{\mathrm{i}0} = 1 = a + \mathrm{i}b$ means $a = 1$ and $b = 0$, in this case it is clear from the diagram that $\alpha = 0$. Therefore c must be equal to zero, so

$$\alpha = \theta.$$

It follows from (1.17) that:

$$a(\theta) = \cos\alpha = \cos\theta, \qquad b(\theta) = \sin\alpha = \sin\theta.$$

Putting them back to (1.16), we obtain again

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta.$$

Note that we have derived the Euler formula without the series expansion. Previously we have derived this formula in a purely algebraic manner. Now we see that $\cos\theta$ and $\sin\theta$ are the cosine and sine functions naturally defined in geometry. This is the unification of algebra and geometry.

It took 250 years for mathematicians to get comfortable with complex numbers. Once fully accepted, the advance of theory of complex variables was rather rapid. In a short span of 40 years, Augustin Louis Cauchy (1789–1857) of France and Georg Friedrich Bernhard Riemann (1826–1866) of Germany developed a beautiful and powerful theory of complex functions, which we will describe in Chap. 2.

In this introductory chapter, we have presented some pieces of historic notes for showing that the logical structure of mathematics is as interesting as any other human endeavor. Now we must leave history behind because of our limited space. For more detailed information, we recommend the following references, from which much of our accounts are taken:

Richard Feynman, Robert B. Leighton, and Mathew Sands, *The Feynman Lectures on Phyics,* Vol. 1, Chapter 22, (1963) Addison Wesley

Eli Maor, *e: the Story of a Number,* (1994) Princeton University Press

Tristan Needham, *Visual Complex Analysis,* Chapter 1, (1997) Oxford University Press

1.6 Polar Form of Complex Numbers

In terms of polar coordinates (r, θ), the variable x and y are

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The complex variable z is then written as

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}. \tag{1.18}$$

The quantity r, known as the modulus, is the absolute value of z and is given by

$$r = |z| = (zz^*)^{1/2} = \left(x^2 + y^2\right)^{1/2}.$$

The angle θ is known as the argument, or phase, of z. Measured in radians, it is given by

$$\theta = \tan^{-1} \frac{y}{x}.$$

If z is in the second or third quadrants, one has to use this equation with care. In the second quadrant, $\tan\theta$ is negative, but in a hand-held calculator, or a computer code, a negative arctangent is interpreted as an angle in the fourth quadrant. In the third quadrant, $\tan\theta$ is positive, but a calculator will interpret a positive arctangent as an angle in the first quadrant. Since an angle is fixed by its sine and cosine, θ is uniquely determined by the pair of equations

$$\cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}.$$

But in practice we usually compute $\tan^{-1}(y/x)$ and adjust for the quadrant problem by adding and subtracting π. Because of its identification as an angle, θ is determined only up to an integer multiple of 2π. We shall make the usual choice of limiting θ to the interval of $0 \leq \theta < 2\pi$ as its principal value. However, in computer codes the principal value is usually chosen in the open interval of $-\pi \leq \theta < \pi$.

Equation (1.18) is called the polar form of z. It is immediately clear that, the complex conjugate of z in the polar form is

$$z^*(r, \theta) = z(r, -\theta) = r\mathrm{e}^{-\mathrm{i}\theta}.$$

In the complex plane, z^* is the reflection of z across the x-axis.

It is helpful to always keep the complex plane in mind. As θ increases, $\mathrm{e}^{\mathrm{i}\theta}$ describes an unit circle in the complex plane as shown in Fig. 1.3. To reach a general complex number z, we must take the unit vector $\mathrm{e}^{\mathrm{i}\theta}$ that points at z and stretch it by the length $|z| = r$.

It is very convenient to multiply or divide two complex numbers in polar forms. Let

$$z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}, \quad z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2},$$

then

$$z_1 z_2 = r_1\mathrm{e}^{\mathrm{i}\theta_1} r_2\mathrm{e}^{\mathrm{i}\theta_2} = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} = r_1 r_2[\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)],$$

$$\frac{z_1}{z_2} = \frac{r_1\mathrm{e}^{\mathrm{i}\theta_1}}{r_2\mathrm{e}^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + \mathrm{i}\sin(\theta_1-\theta_2)].$$

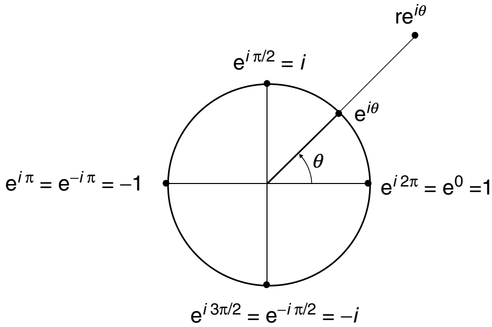


Fig. 1.3. Polar form of complex numbers. The unit circle in the complex plane is described by $\mathrm{e}^{\mathrm{i}\theta}$. A general complex number is given by $r\mathrm{e}^{\mathrm{i}\theta}$

1.6.1 Powers and Roots of Complex Numbers

To obtain the nth power of a complex number, we take the nth power of the modulus and multiply the phase angle by n,

$$z^n = \left(r\mathrm{e}^{\mathrm{i}\theta}\right)^n = r^n \mathrm{e}^{\mathrm{i}n\theta} = r^n \left(\cos n\theta + \mathrm{i}\sin n\theta\right).$$

This is a correct formula for both positive and negative integer n. But if n is a fraction number, we must use this formula with care. For example, we can interpret $z^{1/4}$ as the fourth root of z. In other words, we want to find a number whose 4th power is equal to z. It is instructive to work out the details for the case of $z = 1$. Clearly

$$\begin{aligned} 1^4 &= \left(\mathrm{e}^{\mathrm{i}0}\right)^4 = \mathrm{e}^{\mathrm{i}0} = 1, \\ \mathrm{i}^4 &= \left(\mathrm{e}^{\mathrm{i}\pi/2}\right)^4 = \mathrm{e}^{\mathrm{i}2\pi} = 1, \\ (-1)^4 &= \left(\mathrm{e}^{\mathrm{i}\pi}\right)^4 = \mathrm{e}^{\mathrm{i}4\pi} = 1, \\ (-\mathrm{i})^4 &= \left(\mathrm{e}^{\mathrm{i}3\pi/2}\right)^4 = \mathrm{e}^{\mathrm{i}6\pi} = 1. \end{aligned}$$

Therefore there are four distinct answers

$$1^{1/4} = \begin{cases} 1 \;, \\ \mathrm{i} \;, \\ -1 \;, \\ -\mathrm{i} \;. \end{cases}$$

The multiplicity of roots is tied to the multiple ways of representing 1 in the polar form: $\mathrm{e}^{\mathrm{i}0}$, $\mathrm{e}^{\mathrm{i}2\pi}$, $\mathrm{e}^{\mathrm{i}4\pi}$, etc. Thus to compute all the nth roots of z, we must express z as

$$z = r\mathrm{e}^{\mathrm{i}\theta + \mathrm{i}k2\pi}, \quad (k = 0, 1, 2, \ldots, n-1)$$

and

$$z^{\frac{1}{n}} = \sqrt[n]{r}\mathrm{e}^{\mathrm{i}\theta/n + \mathrm{i}k2\pi/n}, \quad (k = 0, 1, 2, \ldots, n-1).$$

The reason that k stops at $n-1$ is because once k reaches n, $\mathrm{e}^{\mathrm{i}k2\pi/n} = \mathrm{e}^{\mathrm{i}2\pi} = 1$ and the root repeats itself. Therefore there are n distinct roots.

In general, if n and m are positive integers that have no common factor, then

$$z^{m/n} = \sqrt[n]{|z|^m}\mathrm{e}^{\mathrm{i}\frac{m}{n}(\theta + 2k\pi)} = \sqrt[n]{|z|^m}\left[\cos\frac{m}{n}(\theta + 2k\pi) + \mathrm{i}\sin\frac{m}{n}(\theta + 2k\pi)\right]$$

where $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$ and $k = 0, 1, 2, \ldots, n-1$.

Example 1.6.1. Express $(1+\mathrm{i})^8$ in the form of $a + b\mathrm{i}$.

Solution 1.6.1. Let $z = (1+\mathrm{i}) = r\mathrm{e}^{\mathrm{i}\theta}$, where

$$r = (zz^*)^{1/2} = \sqrt{2}, \quad \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}.$$

It follows that:

$$(1+\mathrm{i})^8 = z^8 = r^8\mathrm{e}^{\mathrm{i}8\theta} = 16\mathrm{e}^{\mathrm{i}2\pi} = 16.$$

Example 1.6.2. Express the following in the form of $a + b\mathrm{i}$:

$$\frac{\left(\frac{3}{2}\sqrt{3} + \frac{3}{2}\mathrm{i}\right)^6}{\left(\sqrt{\frac{5}{2}} + \mathrm{i}\sqrt{\frac{5}{2}}\right)^3}.$$

Solution 1.6.2. Let us denote

$$z_1 = \left(\frac{3}{2}\sqrt{3} + \frac{3}{2}\mathrm{i}\right) = r_1\mathrm{e}^{\mathrm{i}\theta_1},$$

$$z_2 = \left(\sqrt{\frac{5}{2}} + \mathrm{i}\sqrt{\frac{5}{2}}\right) = r_2\mathrm{e}^{\mathrm{i}\theta_2},$$

where

$$r_1 = (z_1z_1^*)^{1/2} = 3, \quad \theta_1 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6},$$

$$r_2 = (z_2z_2^*)^{1/2} = \sqrt{5}, \quad \theta_2 = \tan^{-1}(1) = \frac{\pi}{4}.$$

Thus

$$\frac{\left(\frac{3}{2}\sqrt{3}+\frac{3}{2}\mathrm{i}\right)^6}{\left(\sqrt{\frac{5}{2}}+\mathrm{i}\sqrt{\frac{5}{2}}\right)^3}=\frac{z_1^6}{z_2^3}=\frac{\left(3\mathrm{e}^{\mathrm{i}\pi/6}\right)^6}{\left(\sqrt{5}\mathrm{e}^{\mathrm{i}\pi/4}\right)^3}=\frac{3^6\mathrm{e}^{\mathrm{i}\pi}}{\left(\sqrt{5}\right)^3\mathrm{e}^{\mathrm{i}3\pi/4}}$$

$$=\frac{729}{5\sqrt{5}}\mathrm{e}^{\mathrm{i}(\pi-3\pi/4)}=\frac{729}{5\sqrt{5}}\mathrm{e}^{\mathrm{i}\pi/4}$$

$$=\frac{729}{5\sqrt{5}}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)=\frac{729}{5\sqrt{10}}(1+\mathrm{i}).$$

Example 1.6.3. Find all the cube roots of 8.

Solution 1.6.3. Express 8 as a complex number z in the complex plane

$$z=8\mathrm{e}^{\mathrm{i}k2\pi},\quad k=0,1,2,\cdots.$$

Therefore

$$z^{1/3}=(8)^{1/3}\mathrm{e}^{\mathrm{i}k2\pi/3}=2\mathrm{e}^{\mathrm{i}k2\pi/3},\quad k=0,1,2.$$

$$z^{1/3}=\begin{cases}2\mathrm{e}^{\mathrm{i}0}=2, & k=0\\ 2\mathrm{e}^{\mathrm{i}2\pi/3}=2\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)=-1+\mathrm{i}\sqrt{3}, & k=1\\ 2\mathrm{e}^{\mathrm{i}4\pi/3}=2\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right)=-1-\mathrm{i}\sqrt{3}, & k=2.\end{cases}$$

Note that the three roots are on a circle of radius 2 centered at the origin. They are 120° apart.

Example 1.6.4. Find all the cube roots of $\sqrt{2}+\mathrm{i}\sqrt{2}$.

Solution 1.6.4. The polar form of $\sqrt{2}+\mathrm{i}\sqrt{2}$ is

$$z=\sqrt{2}+\mathrm{i}\sqrt{2}=2\mathrm{e}^{\mathrm{i}\pi/4+\mathrm{i}k2\pi}.$$

The cube roots of $\sqrt{2}+\mathrm{i}\sqrt{2}$ are given by

$$z^{1/3}=\begin{cases}(2)^{1/3}\,\mathrm{e}^{\mathrm{i}\pi/12}=(2)^{1/3}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right), & k=0,\\ (2)^{1/3}\,\mathrm{e}^{\mathrm{i}(\pi/12+2\pi/3)}=(2)^{1/3}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right), & k=1,\\ (2)^{1/3}\,\mathrm{e}^{\mathrm{i}(\pi/12+4\pi/3)}=(2)^{1/3}\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right), & k=2.\end{cases}$$

Again the three roots are on a circle 120° apart.

Example 1.6.5. Find all the values of z that satisfy the equation $z^4 = -64$.

Solution 1.6.5. Express -64 as a point in the complex plane

$$-64 = 64\mathrm{e}^{\mathrm{i}\pi + \mathrm{i}k2\pi}, \quad k = 0, 1, 2, \ldots.$$

It follows that:

$$z = (-64)^{1/4} = (64)^{1/4}\,\mathrm{e}^{\mathrm{i}(\pi+2k\pi)/4}, \quad k = 0, 1, 2, 3,$$

$$z = \begin{cases} 2\sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}) = 2 + 2\mathrm{i}, & k = 0 \\ 2\sqrt{2}(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}) = -2 + 2\mathrm{i}, & k = 1 \\ 2\sqrt{2}(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}) = -2 - 2\mathrm{i}, & k = 2 \\ 2\sqrt{2}(\cos\frac{7\pi}{4} + \mathrm{i}\sin\frac{7\pi}{4}) = 2 - 2\mathrm{i}, & k = 3. \end{cases}$$

Note that the four roots are on a circle of radius $\sqrt{8}$ centered at the origin. They are $90°$ apart.

Example 1.6.6. Find all the values of $(1-\mathrm{i})^{3/2}$.

Solution 1.6.6.

$$(1-\mathrm{i}) = \sqrt{2}\mathrm{e}^{\mathrm{i}\theta}, \qquad \theta = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

$$(1-\mathrm{i})^3 = 2\sqrt{2}\mathrm{e}^{\mathrm{i}3\theta + \mathrm{i}k2\pi}, \quad k = 0, 1, 2, \ldots.$$

$$(1-\mathrm{i})^{3/2} = \sqrt[4]{8}\mathrm{e}^{\mathrm{i}(3\theta/2 + k\pi)}, \quad k = 0, 1.$$

$$(1-\mathrm{i})^{3/2} = \begin{cases} \sqrt[4]{8}\left[\cos\left(-\frac{3\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{8}\right)\right], & k = 0 \\ \sqrt[4]{8}\left[\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right], & k = 1. \end{cases}$$

1.6.2 Trigonometry and Complex Numbers

Many trigonometric identities can be most elegantly proved with complex numbers. For example, taking the complex conjugate of the Euler formula

$$(\mathrm{e}^{\mathrm{i}\theta})^* = (\cos\theta + \mathrm{i}\sin\theta)^*,$$

we have

$$\mathrm{e}^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta.$$

It is interesting to write this equation as

$$\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(-\theta)} = \cos(-\theta) + \mathrm{i}\sin(-\theta).$$

Comparing the last two equations, we find that

$$\begin{aligned}\cos(-\theta) &= \cos\theta \\ \sin(-\theta) &= -\sin\theta\end{aligned}$$

which is consistent with what we know about the cosine and sine functions of trigonometry.

Adding and subtracting $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$, we have

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} &= (\cos\theta + \mathrm{i}\sin\theta) + (\cos\theta - \mathrm{i}\sin\theta) = 2\cos\theta, \\ \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} &= (\cos\theta + \mathrm{i}\sin\theta) - (\cos\theta - \mathrm{i}\sin\theta) = 2\mathrm{i}\sin\theta.\end{aligned}$$

Using them one can easily express the powers of cosine and sine in terms of $\cos n\theta$ and $\sin n\theta$. For example, with $n = 2$

$$\begin{aligned}\cos^2\theta &= \left[\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^2 = \frac{1}{4}\left(\mathrm{e}^{\mathrm{i}2\theta} + 2\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) \\ &= \frac{1}{2}\left[\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + \mathrm{e}^{\mathrm{i}0}\right] = \frac{1}{2}\left(\cos 2\theta + 1\right),\end{aligned}$$

$$\begin{aligned}\sin^2\theta &= \left[\frac{1}{2i}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^2 = -\frac{1}{4}\left(\mathrm{e}^{\mathrm{i}2\theta} - 2\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) \\ &= \frac{1}{2}\left[-\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + \mathrm{e}^{\mathrm{i}0}\right] = \frac{1}{2}\left(-\cos 2\theta + 1\right).\end{aligned}$$

To find an identity for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$, we can view them as components of $\exp[\mathrm{i}(\theta_1 + \theta_2)]$. Since

$$\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2} = \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2),$$

and

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2} &= [\cos\theta_1 + \mathrm{i}\sin\theta_1][\cos\theta_2 + \mathrm{i}\sin\theta_2] \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}\,(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2),\end{aligned}$$

equating the real and imaginary parts of these equivalent expressions, we get the familiar formulas

$$\begin{aligned}\cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \\ \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.\end{aligned}$$

From these two equations, it follows that:

$$\tan(\theta_1 + \theta_2) = \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} = \frac{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}.$$

Dividing top and bottom by $\cos\theta_1\cos\theta_2$, we obtain

$$\tan(\theta_1+\theta_2)=\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}.$$

This formula can be derived directly with complex numbers. Let z_1 and z_2 be two points in the complex plane whose x components are both equal to 1.

$$z_1=1+\mathrm{i}y_1=r_1\mathrm{e}^{\mathrm{i}\theta_1},\qquad \tan\theta_1=\frac{y_1}{1}=y_1,$$
$$z_2=1+\mathrm{i}y_2=r_2\mathrm{e}^{\mathrm{i}\theta_2},\qquad \tan\theta_2=\frac{y_2}{1}=y_2.$$

The product of the two is given by

$$z_1z_2=r_1r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)},\qquad \tan(\theta_1+\theta_2)=\frac{\mathrm{Im}(z_1z_2)}{\mathrm{Re}(z_1z_2)}.$$

But

$$z_1z_2=(1+\mathrm{i}y_1)(1+\mathrm{i}y_2)=(1-y_1y_2)+\mathrm{i}(y_1+y_2),$$

therefore

$$\tan(\theta_1+\theta_2)=\frac{\mathrm{Im}(z_1z_2)}{\mathrm{Re}(z_1z_2)}=\frac{y_1+y_2}{1-y_1y_2}=\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}.$$

These identities can, of course, be demonstrated geometrically. However, it is much easier to prove them algebraically with complex numbers.

Example 1.6.7. Prove De Moivre formula

$$(\cos\theta+\mathrm{i}\sin\theta)^n=\cos n\theta+\mathrm{i}\sin n\theta.$$

Solution 1.6.7. Since $(\cos\theta+\mathrm{i}\sin\theta)=\mathrm{e}^{\mathrm{i}\theta}$, it follows: that

$$\begin{aligned}(\cos\theta+\mathrm{i}\sin\theta)^n&=\left(\mathrm{e}^{\mathrm{i}\theta}\right)^n=\mathrm{e}^{\mathrm{i}n\theta}\\&=\cos n\theta+\mathrm{i}\sin n\theta.\end{aligned}$$

This theorem was published in 1707 by Abraham De Moivre, a French mathematician working in London.

Example 1.6.8. Use De Moivre's theorem and binomial expansion to express $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Solution 1.6.8.

$$\begin{aligned}
\cos 4\theta + \mathrm{i}\sin 4\theta &= \mathrm{e}^{\mathrm{i}4\theta} = \left(\mathrm{e}^{\mathrm{i}\theta}\right)^4 = (\cos\theta + \mathrm{i}\sin\theta)^4 \\
&= \cos^4\theta + 4\cos^3\theta(\mathrm{i}\sin\theta) + 6\cos^2\theta(\mathrm{i}\sin\theta)^2 \\
&\quad + 4\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4 \\
&= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) \\
&\quad + \mathrm{i}\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)
\end{aligned}$$

Equating the real and imaginary parts of these complex expressions, we obtain

$$\begin{aligned}
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \\
\sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.
\end{aligned}$$

Example 1.6.9. Express $\cos^4\theta$ and $\sin^4\theta$ in terms of multiples of θ.

Solution 1.6.9.

$$\begin{aligned}
\cos^4\theta &= \left[\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^4 \\
&= \frac{1}{16}\left[\left(\mathrm{e}^{\mathrm{i}4\theta} + \mathrm{e}^{-\mathrm{i}4\theta}\right) + 4\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + 6\right] \\
&= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.
\end{aligned}$$

$$\begin{aligned}
\sin^4\theta &= \left[\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^4 \\
&= \frac{1}{16}\left[\left(\mathrm{e}^{\mathrm{i}4\theta} + \mathrm{e}^{-\mathrm{i}4\theta}\right) - 4\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + 6\right] \\
&= \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}.
\end{aligned}$$

Example 1.6.10. Show that

$$\begin{aligned}
\cos(\theta_1 + \theta_2 + \theta_3) &= \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3 \\
&\quad - \sin\theta_1\sin\theta_3\cos\theta_2 - \sin\theta_2\sin\theta_3\cos\theta_1, \\
\sin(\theta_1 + \theta_2 + \theta_3) &= \sin\theta_1\cos\theta_2\cos\theta_3 + \sin\theta_2\cos\theta_1\cos\theta_3 \\
&\quad + \sin\theta_3\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\sin\theta_3.
\end{aligned}$$

Solution 1.6.10.

$$\cos(\theta_1+\theta_2+\theta_3)+\mathrm{i}\sin(\theta_1+\theta_2+\theta_3)=\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2+\theta_3)}=\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}\mathrm{e}^{\mathrm{i}\theta_3},$$

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}\mathrm{e}^{\mathrm{i}\theta_3} &= (\cos\theta_1+\mathrm{i}\sin\theta_1)(\cos\theta_2+\mathrm{i}\sin\theta_2)(\cos\theta_3+\mathrm{i}\sin\theta_3)\\ &= \cos\theta_1(1+\mathrm{i}\tan\theta_1)\cos\theta_2(1+\mathrm{i}\tan\theta_2)\cos\theta_3(1+\mathrm{i}\tan\theta_3).\end{aligned}$$

Since

$$(1+a)(1+b)(1+c)=1+(a+b+b)+(ab+bc+ca)+abc,$$

$$\begin{aligned}&(1+\mathrm{i}\tan\theta_1)(1+\mathrm{i}\tan\theta_2)(1+\mathrm{i}\tan\theta_3)=1+\mathrm{i}(\tan\theta_1+\tan\theta_2+\tan\theta_3)\\ &\quad+\mathrm{i}^2(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)+\mathrm{i}^3\tan\theta_1\tan\theta_2\tan\theta_3\\ &\qquad=[1-(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)]\\ &\qquad\quad+\mathrm{i}[(\tan\theta_1+\tan\theta_2+\tan\theta_3)-\tan\theta_1\tan\theta_2\tan\theta_3].\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}\mathrm{e}^{\mathrm{i}\theta_3} &= \cos\theta_1\cos\theta_2\cos\theta_3\\ &\quad\times\left\{\begin{array}{l}[1-(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)]\\ +\mathrm{i}[(\tan\theta_1+\tan\theta_2+\tan\theta_3)-\tan\theta_1\tan\theta_2\tan\theta_3]\end{array}\right\}.\end{aligned}$$

Equating the real and imaginary parts

$$\begin{aligned}\cos(\theta_1+\theta_2+\theta_3) &= \cos\theta_1\cos\theta_2\cos\theta_3\\ &\quad\times[1-(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)]\\ &= \cos\theta_1\cos\theta_2\cos\theta_3-\sin\theta_1\sin\theta_2\cos\theta_3\\ &\quad-\sin\theta_1\sin\theta_3\cos\theta_2-\sin\theta_2\sin\theta_3\cos\theta_1,\end{aligned}$$

$$\begin{aligned}\sin(\theta_1+\theta_2+\theta_3) &= \cos\theta_1\cos\theta_2\cos\theta_3\\ &\quad\times[(\tan\theta_1+\tan\theta_2+\tan\theta_3)-\tan\theta_1\tan\theta_2\tan\theta_3]\\ &= \sin\theta_1\cos\theta_2\cos\theta_3+\sin\theta_2\cos\theta_1\cos\theta_3\\ &\quad+\sin\theta_3\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\sin\theta_3.\end{aligned}$$

Example 1.6.11. If $\theta_1,\theta_2,\theta_3$ are the three interior angles of a triangle, show that

$$\tan\theta_1+\tan\theta_2+\tan\theta_3=\tan\theta_1\tan\theta_2\tan\theta_3.$$

Solution 1.6.11. Since

$$\tan(\theta_1+\theta_2+\theta_3)=\frac{\sin(\theta_1+\theta_2+\theta_3)}{\cos(\theta_1+\theta_2+\theta_3)},$$

using the results of the previous problem and dividing the top and bottom by $\cos\theta_1\cos\theta_2\cos\theta_3$, we have

$$\tan(\theta_1+\theta_2+\theta_3)=\frac{\tan\theta_1+\tan\theta_2+\tan\theta_3-\tan\theta_1\tan\theta_2\tan\theta_3}{1-\tan\theta_1\tan\theta_2-\tan\theta_2\tan\theta_3-\tan\theta_3\tan\theta_1}.$$

Now $\theta_1,\theta_2,\theta_3$ are the three interior angles of a triangle, so $\theta_1+\theta_2+\theta_3=\pi$ and $\tan(\theta_1+\theta_2+\theta_3)=\tan\pi=0$. Therefore

$$\tan\theta_1+\tan\theta_2+\tan\theta_3=\tan\theta_1\tan\theta_2\tan\theta_3.$$

Example 1.6.12. Show that

$$\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2n-1)\theta=\frac{\sin n\theta\cos n\theta}{\sin\theta},$$
$$\sin\theta+\sin 3\theta+\sin 5\theta+\cdots+\sin(2n-1)\theta=\frac{\sin^2 n\theta}{\sin\theta}.$$

Solution 1.6.12. Let

$$C=\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2n-1)\theta,$$
$$S=\sin\theta+\sin 3\theta+\sin 5\theta+\cdots+\sin(2n-1)\theta.$$

$$\begin{aligned}Z=C+\mathrm{i}S&=(\cos\theta+\mathrm{i}\sin\theta)+(\cos 3\theta+\mathrm{i}\sin 3\theta)\\&\quad+(\cos 5\theta+\mathrm{i}\sin 5\theta)+\cdots+(\cos(n-1)\theta+\mathrm{i}\sin(2n-1)\theta)\\&=\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}3\theta}+\mathrm{e}^{\mathrm{i}5\theta}+\cdots+\mathrm{e}^{\mathrm{i}(2n-1)\theta}.\end{aligned}$$

$$\mathrm{e}^{\mathrm{i}2\theta}Z=\mathrm{e}^{\mathrm{i}3\theta}+\mathrm{e}^{\mathrm{i}5\theta}+\cdots+\mathrm{e}^{\mathrm{i}(2n-1)\theta}+\mathrm{e}^{\mathrm{i}(2n+1)\theta},$$

$$Z-\mathrm{e}^{\mathrm{i}2\theta}Z=\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}(2n+1)\theta},$$

$$\begin{aligned}Z&=\frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}(2n+1)\theta}}{1-\mathrm{e}^{\mathrm{i}2\theta}}=\frac{\mathrm{e}^{\mathrm{i}\theta}(1-\mathrm{e}^{\mathrm{i}2n\theta})}{\mathrm{e}^{\mathrm{i}\theta}\left(\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\right)}=\frac{\mathrm{e}^{in\theta}\left(\mathrm{e}^{-in\theta}-\mathrm{e}^{in\theta}\right)}{\left(\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\right)}\\&=\frac{\mathrm{e}^{in\theta}\sin n\theta}{\sin\theta}=\frac{\cos n\theta\sin n\theta}{\sin\theta}+i\frac{\sin n\theta\sin n\theta}{\sin\theta}.\end{aligned}$$

Therefore

$$C=\frac{\cos n\theta\sin n\theta}{\sin\theta},\qquad S=\frac{\sin^2 n\theta}{\sin\theta}.$$

Example 1.6.13. For $r < 1$, show that

$$\left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2 = \frac{1}{1 - 2r^2 \cos\theta + r^4}.$$

Solution 1.6.13. Let

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} r^{2n} \cos n\theta + \mathrm{i} \sum_{n=0}^{\infty} r^{2n} \sin n\theta \\ &= \sum_{n=0}^{\infty} r^{2n} (\cos n\theta + \mathrm{i} \sin n\theta) = \sum_{n=0}^{\infty} r^{2n} \mathrm{e}^{\mathrm{i}n\theta} \\ &= 1 + r^2 \mathrm{e}^{\mathrm{i}\theta} + r^4 \mathrm{e}^{\mathrm{i}2\theta} + r^6 \mathrm{e}^{\mathrm{i}3\theta} + \cdots. \end{aligned}$$

Since $r < 1$, so this is a convergent series

$$r^2 \mathrm{e}^{\mathrm{i}\theta} Z = r^2 \mathrm{e}^{\mathrm{i}\theta} + r^4 \mathrm{e}^{\mathrm{i}2\theta} + r^6 \mathrm{e}^{\mathrm{i}3\theta} + \cdots.$$

$$Z - r^2 \mathrm{e}^{\mathrm{i}\theta} Z = 1,$$

$$Z = \frac{1}{1 - r^2 \mathrm{e}^{\mathrm{i}\theta}}.$$

$$\begin{aligned} |Z|^2 = ZZ^* &= \frac{1}{1 - r^2 \mathrm{e}^{\mathrm{i}\theta}} \times \frac{1}{1 - r^2 \mathrm{e}^{-\mathrm{i}\theta}} \\ &= \frac{1}{1 - r^2 (\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) + r^4} = \frac{1}{1 - 2r^2 \cos\theta + r^4}, \end{aligned}$$

But

$$\begin{aligned} |Z|^2 &= \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta + \mathrm{i} \sum_{n=0}^{\infty} r^{2n} \sin n\theta\right) \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta - \mathrm{i} \sum_{n=0}^{\infty} r^{2n} \sin n\theta\right) \\ &= \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2. \end{aligned}$$

Therefore

$$\left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2 = \frac{1}{1 - 2r^2 \cos\theta + r^4}.$$

This is the intensity of the light, transmitted through a film after multiple reflections at the surfaces of the film and r is the fraction of light reflected each time.

1.6.3 Geometry and Complex Numbers

There are three geometric representations of the complex number $z = x + \mathrm{i}y$:

(a) as the point $P\,(x, y)$ in the xy plane,

(b) as the vector OP from the origin to the point P,

(c) as any vector that is of the same length and same direction as OP.

For example, $z_A = 3 + \mathrm{i}$ can be represented by the point A in Fig. 1.4. It can also be represented by the vector z_A. Similarly $z_B = -2 + 3\mathrm{i}$ can be represented by the point B and the vector z_B. Now let us define z_C as $z_A + z_B$,

$$z_C = z_A + z_B = (3 + \mathrm{i}) + (-2 + 3\mathrm{i}) = 1 + 4\mathrm{i}.$$

So z_C is represented by the point C and the vector z_C. Clearly the two shaded triangles in Fig. 1.4 are identical. The vector AC (from A to C) is not only parallel to z_B, it is also of the same length as z_B. In this sense, we say that the vector AC can also represent z_B. Thus z_A, z_B, and $z_A + z_B$ are three sides of the triangle OAC. Since the sum of two sides of a triangle must be greater or equal to the third side, it follows that

$$|z_A| + |z_B| \geq |z_A + z_B|\,.$$

Since $z_B = z_C - z_A$, and z_B is the same as the AC, we can interpret $z_C - z_A$ as the vector from the tip of z_A to the tip of z_C. The distance between C and A is simply $|z_C - z_A|$.

If z is a variable and z_A is fixed, then a circle of radius r centered at z_A is described by the equation

$$|z - z_A| = r.$$

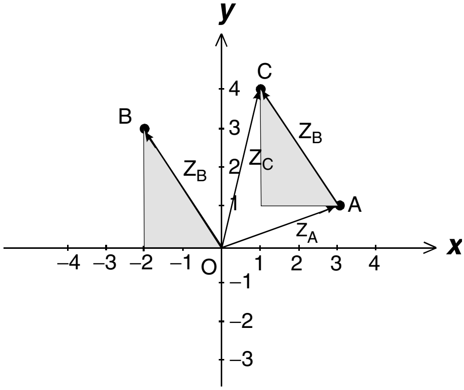


Fig. 1.4. Addition and subtraction of complex numbers in the complex plane. A complex number can be represented by a point in the complex plane, or by the vector from the origin to that point. The vector can be moved parallel to itself

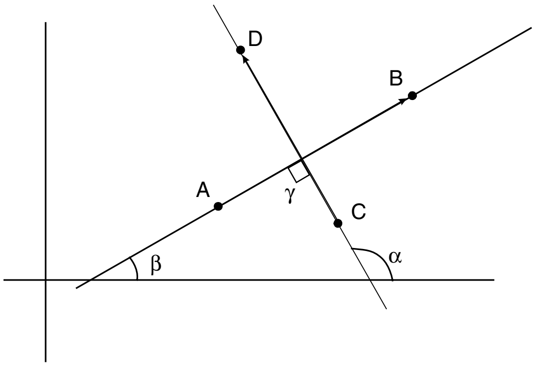


Fig. 1.5. Perpendicular segments. If AB and CD are perpendicular, then the ratio of $z_B - z_A$ and $z_D - z_C$ must be purely imaginary

If the two segments AB and CD are parallel, then

$$z_B - z_A = k\,(z_D - z_C)\,,$$

where k is a real number. If $k = 1$, then A, B, C, D must be the vertices of a parallelogram.

If the two segments AB and CD are perpendicular to each other, then the ratio $\frac{z_D - z_C}{z_B - z_A}$ must be a pure imaginary number. This can be seen as follows.

The segment AB in Fig. 1.5 can be expressed as

$$z_B - z_A = |z_B - z_A|\,\mathrm{e}^{\mathrm{i}\beta},$$

and segment CD as

$$z_D - z_C = |z_D - z_C|\,\mathrm{e}^{\mathrm{i}\alpha}.$$

So

$$\frac{z_D - z_C}{z_B - z_A} = \frac{|z_D - z_C|\,\mathrm{e}^{\mathrm{i}\alpha}}{|z_B - z_A|\,\mathrm{e}^{\mathrm{i}\beta}} = \frac{|z_D - z_C|}{|z_B - z_A|}\mathrm{e}^{\mathrm{i}(\alpha-\beta)}.$$

It is well known that the exterior angle is equal to the sum of the two interior angles, that is, in Fig. 1.5 $\alpha = \beta + \gamma$, or $\gamma = \alpha - \beta$. If AB is perpendicular to CD, then $\gamma = \frac{\pi}{2}$, and

$$\mathrm{e}^{\mathrm{i}(\alpha-\beta)} = \mathrm{e}^{\mathrm{i}\gamma} = \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}.$$

Thus

$$\frac{z_D - z_C}{z_B - z_A} = \frac{|z_D - z_C|}{|z_B - z_A|}\mathrm{i}.$$

Since $\frac{|z_D - z_C|}{|z_B - z_A|}$ is real, so $\frac{z_D - z_C}{z_B - z_A}$ must be imaginary.

The following examples will illustrate how to use these principles to solve problems in geometry.

Example 1.6.14. Determine the curve in the complex plane that is described by

$$\left|\frac{z+1}{z-1}\right| = 2.$$

Solution 1.6.14. $\left|\frac{z+1}{z-1}\right| = 2$ can be written as $|z+1| = 2\,|z-1|$. With $z = x + \mathrm{i}y$, this equation becomes

$$|(x+1) + \mathrm{i}y| = 2\,|(x-1) + \mathrm{i}y|,$$

$$\{[(x+1) + \mathrm{i}y]\,[(x+1) - \mathrm{i}y]\}^{1/2} = 2\,\{[(x-1) + \mathrm{i}y]\,[(x-1) - \mathrm{i}y]\}^{1/2}.$$

Square both sides

$$(x+1)^2 + y^2 = 4\,(x-1)^2 + 4y^2.$$

This gives

$$3x^2 - 10x + 3y^2 + 3 = 0,$$

which can be written as

$$\left(x - \frac{5}{3}\right)^2 + y^2 - \left(\frac{5}{3}\right)^2 + 1 = 0,$$

or

$$\left(x - \frac{5}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2.$$

This represents a circle of radius $\frac{4}{3}$ with a center at $\left(\frac{5}{3}, 0\right)$.

Example 1.6.15. In the parallelogram shown in Fig. 1.6, the base is fixed along the x-axis and is of length a. The length of the other side is b. As the angle θ between the two sides changes, determine the locus of the center of the parallelogram.

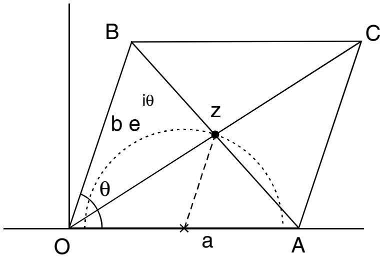


Fig. 1.6. The curve described by the center of a parallelogram. If the base is fixed, the locus of the center is a circle

Solution 1.6.15. Let the origin of the coordinates be at the left bottom corner of the parallelogram. So

$$z_A = a, \qquad z_B = b\mathrm{e}^{\mathrm{i}\theta}.$$

Let the center of the parallelogram be z which is at the midpoint of the diagonal OC. Thus

$$z = \frac{1}{2}\left(z_A + z_B\right) = \frac{1}{2}a + \frac{1}{2}b\mathrm{e}^{\mathrm{i}\theta}$$

or

$$z - \frac{1}{2}a = \frac{1}{2}b\mathrm{e}^{\mathrm{i}\theta}$$

It follows that:

$$\left|z - \frac{1}{2}a\right| = \left|\frac{1}{2}b\mathrm{e}^{\mathrm{i}\theta}\right| = \frac{1}{2}b.$$

Therefore the locus of the center is a circle of radius $\frac{1}{2}b$ centered at $\frac{1}{2}a$. Half of the circle is shown in Fig. 1.6.

Example 1.6.16. If E, F, G, H are midpoints of the quadrilateral $ABCD$. Prove that $EFGH$ is a parallelogram.

Solution 1.6.16. Let the vector from origin to any point P be z_P, then from Fig. 1.7 we see that

$$z_E = z_A + \frac{1}{2}\left(z_B - z_A\right),$$
$$z_F = z_B + \frac{1}{2}\left(z_C - z_B\right),$$

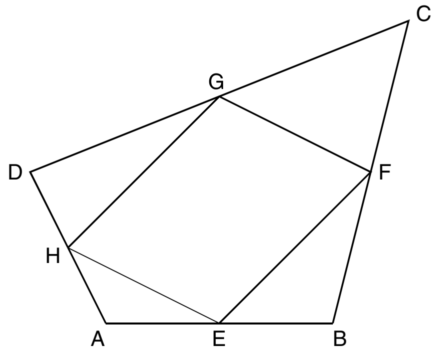


Fig. 1.7. Parallelogram formed by the midpoints of a quadrilateral

$$z_F - z_E = z_B + \frac{1}{2}(z_C - z_B) - z_A - \frac{1}{2}(z_B - z_A) = \frac{1}{2}(z_C - z_A).$$

$$\begin{aligned} z_G &= z_D + \frac{1}{2}(z_C - z_D), \\ z_H &= z_A + \frac{1}{2}(z_D - z_A), \end{aligned}$$

$$z_G - z_H = z_D + \frac{1}{2}(z_C - z_D) - z_A - \frac{1}{2}(z_D - z_A) = \frac{1}{2}(z_C - z_A).$$

Thus

$$z_F - z_E = z_G - z_H.$$

Therefore $EFGH$ is a parallelogram.

Example 1.6.17. Use complex number to show that the diagonals of a rhombus (a parallelogram with equal sides) are perpendicular to each other, as shown in Fig. 1.8.

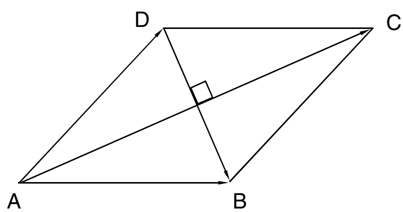


Fig. 1.8. The diagonals of a rhombus are perpendicular to each other

Solution 1.6.17. The diagonal AC is given by $z_C - z_A$, and the diagonal DB is given by $z_B - z_D$. Let the length of each side of the rhombus be a, and the origin of the coordinates coincide with A. Furthermore let the x-axis be along the line AB. Thus

$$z_A = 0, \quad z_B = a, \quad z_D = a\mathrm{e}^{\mathrm{i}\theta}.$$

Furthermore,

$$z_C = z_B + z_D = a + a\mathrm{e}^{\mathrm{i}\theta}.$$

Therefore

$$\begin{aligned} z_C - z_A &= a + a\mathrm{e}^{\mathrm{i}\theta} = a\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right), \\ z_B - z_D &= a - a\mathrm{e}^{\mathrm{i}\theta} = a\left(1 - \mathrm{e}^{\mathrm{i}\theta}\right). \end{aligned}$$

Thus

$$\begin{aligned} \frac{z_C - z_A}{z_B - z_D} &= \frac{a\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right)}{a\left(1 - \mathrm{e}^{\mathrm{i}\theta}\right)} = \frac{\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right)\left(1 - \mathrm{e}^{-\mathrm{i}\theta}\right)}{\left(1 - \mathrm{e}^{\mathrm{i}\theta}\right)\left(1 - \mathrm{e}^{-\mathrm{i}\theta}\right)} \\ &= \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2 - \left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right)} = \mathrm{i}\frac{\sin\theta}{1 - \cos\theta}. \end{aligned}$$

Since $\frac{\sin\theta}{1-\cos\theta}$ is real, $\frac{z_C - z_A}{z_D - z_B}$ is purely imaginary. Hence AC is perpendicular to DB.

Example 1.6.18. In the triangle AOB, shown in Fig. 1.9, the angle between AO and OB is 90° and the length of AO is the same as the length of OB. The point D trisects the line AB such that $AD = 2DB$, and C is the midpoint of OB. Show that AC is perpendicular to OD.

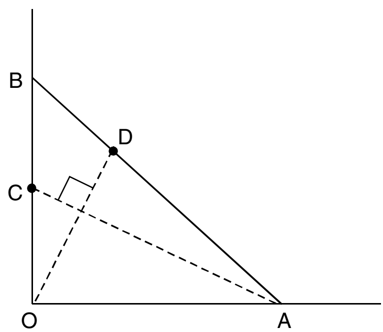


Fig. 1.9. A problem in geometry. If OA is perpendicular to OB and $OA = OB$, then the line CA is perpendicular to the line OD where C is the midpoint of OB and $DA = 2BD$

Solution 1.6.18. Let the real axis be along OA and the imaginary axis along OB. Let the length of OA and OB be a. Thus

$$z_O = 0, \quad z_A = a, \quad z_B = a\mathrm{i}, \quad z_C = \frac{1}{2}a\mathrm{i},$$
$$z_D = z_A + \frac{2}{3}(z_B - z_A) = a + \frac{2}{3}(a\mathrm{i} - a) = \frac{1}{3}a(1+2\mathrm{i}),$$
$$z_D - z_O = \frac{1}{3}a(1+2\mathrm{i}) - 0 = \frac{1}{3}a(1+2\mathrm{i}).$$

The vector AC is given by $z_C - z_A$,

$$z_C - z_A = \frac{1}{2}a\mathrm{i} - a = \mathrm{i}\frac{1}{2}a(1+2\mathrm{i}).$$

Thus

$$\frac{z_C - z_A}{z_D - z_O} = \mathrm{i}\frac{3}{2}.$$

Since this is purely imaginary, therefore AC is perpendicular to OD.

1.7 Elementary Functions of Complex Variable

1.7.1 Exponential and Trigonometric Functions of z

The exponential function e^z is of fundamental importance, not only for its own sake, but also as a basis for defining all the other elementary functions. The exponential function of real variable is well known. Now we wish to give meaning to e^z when $z = x + \mathrm{i}y$. In the spirit of Euler, we can work our way in a purely manipulative manner. Assuming that e^z obeys all the familiar rules of the exponential function of a real number, we have

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x\mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y). \tag{1.19}$$

Thus we can define e^z as $\mathrm{e}^x(\cos y + \mathrm{i}\sin y)$. It reduces to e^x when the imaginary part of z vanishes. It is also easy to show that

$$\mathrm{e}^{z_1}\mathrm{e}^{z_2} = \mathrm{e}^{z_1+z_2}.$$

Furthermore, in Chap. 2 we shall consider in detail the meaning of derivatives with respect to a complex z. Now it suffices to know that

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \mathrm{e}^z.$$

Therefore the definition of (1.19) preserves all the familiar properties of the exponential function.

We have already seen that

$$\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}), \qquad \sin\theta = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}).$$

On the basis of these equations, we extend the definitions of the cosine and sine into the complex domain. Thus we define

$$\cos z = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}), \qquad \sin z = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}).$$

The rest of the trigonometric functions of z are defined in a usual way. For example,

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}, \\ \sec z &= \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}. \end{aligned}$$

With these definitions we can show that all the familiar formulas of trigonometry remain valid when real variable x is replaced by complex variable z:

$$\cos(-z) = \cos z, \qquad \sin(-z) = -\sin z,$$

$$\cos^2 z + \sin^2 z = 1,$$

$$\begin{aligned} \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, \\ \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \\ \frac{\mathrm{d}}{\mathrm{d}z}\cos z &= -\sin z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z. \end{aligned}$$

To prove them, we must start with their definitions. For example,

$$\begin{aligned} \cos^2 z + \sin^2 z &= \left[\frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})\right]^2 + \left[\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})\right]^2 \\ &= \frac{1}{4}(\mathrm{e}^{\mathrm{i}2z} + 2 + \mathrm{e}^{-\mathrm{i}2z}) - \frac{1}{4}(\mathrm{e}^{\mathrm{i}2z} - 2 + \mathrm{e}^{-\mathrm{i}2z}) = 1. \end{aligned}$$

Example 1.7.1. Express $\mathrm{e}^{1-\mathrm{i}}$ in the form of $a + b\mathrm{i}$, accurate to three decimal places.

Solution 1.7.1.

$$\mathrm{e}^{1-\mathrm{i}} = \mathrm{e}^{1}\mathrm{e}^{-\mathrm{i}} = e\left(\cos 1 - \mathrm{i}\sin 1\right).$$

Using a hand-held calculator, we find

$$\begin{aligned} \mathrm{e}^{1-\mathrm{i}} &\simeq 2.718(0.5403 - 0.8415\mathrm{i}) \\ &= 1.469 - 2.287\mathrm{i}. \end{aligned}$$

Example 1.7.2. Show that

$$\sin 2z = 2 \sin z \cos z.$$

Solution 1.7.2.

$$\begin{aligned} 2 \sin z \cos z &= 2\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})\frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \\ &= \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}2z} - \mathrm{e}^{-\mathrm{i}2z}) = \sin 2z. \end{aligned}$$

Example 1.7.3. Compute $\sin(1 - \mathrm{i})$.

Solution 1.7.3. By definition

$$\begin{aligned} \sin(1-\mathrm{i}) &= \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}(1-\mathrm{i})} - \mathrm{e}^{-\mathrm{i}(1-\mathrm{i})}\right) = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{1+\mathrm{i}} - \mathrm{e}^{-1-\mathrm{i}}\right) \\ &= \frac{1}{2\mathrm{i}}\left\{\mathrm{e}\left[\cos(1) + \mathrm{i}\sin(1)\right]\right\} - \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{-1}\left[\cos(1) - \mathrm{i}\sin(1)\right]\right\} \\ &= \frac{1}{2\mathrm{i}}\left(\mathrm{e} - \mathrm{e}^{-1}\right)\cos(1) + \frac{1}{2}\left(\mathrm{e} + \mathrm{e}^{-1}\right)\sin(1). \end{aligned}$$

We can get the same result by using the trigonometric addition formula.

1.7.2 Hyperbolic Functions of z

The following particular combinations of exponentials arise frequently,

$$\cosh z = \frac{1}{2}\left(\mathrm{e}^{z} + \mathrm{e}^{-z}\right), \qquad \sinh z = \frac{1}{2}\left(\mathrm{e}^{z} - \mathrm{e}^{-z}\right).$$

They are called hyperbolic cosine (*abbreviated* cosh) and hyperbolic sine (*abbreviated* sinh). Clearly

$$\cosh(-z) = \cosh z, \qquad \sinh(-z) = -\sinh z.$$

The other hyperbolic functions are defined in a similar way to parallel the trigonometric functions:

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{1}{\tanh z}, \\ \sec hz &= \frac{1}{\cosh z}, \qquad \csc hz = \frac{1}{\sinh z}. \end{aligned}$$

With these definitions, all identities involving hyperbolic functions of real variable are preserved when the variable is complex. For example,

$$\cosh^2 z - \sinh^2 z = \frac{1}{4}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right)^2 - \frac{1}{4}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right)^2 = 1,$$

$$\begin{aligned}\sinh 2z &= \frac{1}{2}\left(\mathrm{e}^{2z} - \mathrm{e}^{-2z}\right) = \frac{1}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right)\left(\mathrm{e}^z + \mathrm{e}^{-z}\right)\\ &= 2\sinh z \cosh z.\end{aligned}$$

There is a close relationship between the trigonometric and hyperbolic functions when the variable is complex. For example,

$$\begin{aligned}\sin \mathrm{i}z &= \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}(\mathrm{i}z)} - \mathrm{e}^{-\mathrm{i}(\mathrm{i}z)}\right) = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{-z} - \mathrm{e}^{z}\right)\\ &= \frac{\mathrm{i}}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right) = \mathrm{i}\sinh z.\end{aligned}$$

Similarly we can show

$$\cos \mathrm{i}z = \cosh z,$$

$$\sinh \mathrm{i}z = \mathrm{i}\sin z, \quad \cosh \mathrm{i}z = \cos z.$$

Furthermore,

$$\begin{aligned}\sin z &= \sin(x + \mathrm{i}y) = \sin x \cos \mathrm{i}y + \cos x \sin \mathrm{i}y\\ &= \sin x \cosh y + \mathrm{i}\cos x \sinh y,\\ \cos z &= \cos x \cosh y - \mathrm{i}\sin x \sinh y.\end{aligned}$$

Example 1.7.4. Show that

$$\frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \cosh z.$$

Solution 1.7.4.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\cosh z &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right) = \frac{1}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right) = \sinh z,\\ \frac{\mathrm{d}}{\mathrm{d}z}\sinh z &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right) = \frac{1}{2}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right) = \cosh z.\end{aligned}$$

Example 1.7.5. Evaluate $\cos(1 + 2\mathrm{i})$.

Solution 1.7.5.

$$\begin{aligned}\cos(1 + 2\mathrm{i}) &= \cos 1 \cosh 2 - \mathrm{i}\sin 1 \sinh 2\\ &= (0.5403)\,(3.7622) - \mathrm{i}\,(0.8415)\,(3.6269) = 2.033 - 3.052\mathrm{i}.\end{aligned}$$

Example 1.7.6. Evaluate $\cos(\pi - \mathrm{i})$.

Solution 1.7.6. By definition,

$$\begin{aligned}\cos(\pi - \mathrm{i}) &= \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}(\pi-\mathrm{i})} + \mathrm{e}^{-\mathrm{i}(\pi-\mathrm{i})}\right) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\pi+1} + \mathrm{e}^{-\mathrm{i}\pi-1}\right) \\ &= \frac{1}{2}(-\mathrm{e} - \mathrm{e}^{-1}) = -\cosh(1) = -1.543.\end{aligned}$$

We get the same result by the expansion,

$$\begin{aligned}\cos(\pi - \mathrm{i}) &= \cos\pi\cosh(1) + \mathrm{i}\sin\pi\sinh(1) \\ &= -\cosh(1) = -1.543.\end{aligned}$$

1.7.3 Logarithm and General Power of z

The natural logarithm of $z = x + \mathrm{i}y$ is denoted $\ln z$ and is defined in a similar way as in the real variable, namely as the inverse of the exponential function. However, there is an important difference. A real valued exponential $y = \mathrm{e}^x$ is a one to one function, since two different x always produce two different values of y. Strictly speaking, only one to one function has an inverse, because only then will each value of y can be the image of exactly one x value. But the complex exponential e^z is a multivalued function, since

$$\mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y).$$

When y is increased by an integer multiple of 2π, the exponential returns to its original value. Therefore to define a complex logarithm we have to relax the one to one restriction. Thus,

$$w = \ln z$$

is defined for $z \neq 0$ by the relation

$$\mathrm{e}^w = z.$$

If we set

$$w = u + \mathrm{i}v, \qquad z = r\mathrm{e}^{\mathrm{i}\theta},$$

this becomes

$$\mathrm{e}^w = \mathrm{e}^{u+\mathrm{i}v} = \mathrm{e}^u\mathrm{e}^{\mathrm{i}v} = r\mathrm{e}^{\mathrm{i}\theta}.$$

Since

$$|\mathrm{e}^w| = \left[(\mathrm{e}^w)(\mathrm{e}^w)^*\right]^{1/2} = \left(\mathrm{e}^{u+\mathrm{i}v}\mathrm{e}^{u-\mathrm{i}v}\right)^{1/2} = \mathrm{e}^u,$$

$$|e^w| = \left[\left(re^{i\theta}\right)\left(re^{i\theta}\right)^*\right]^{1/2} = \left[\left(re^{i\theta}\right)\left(re^{-i\theta}\right)\right]^{1/2} = r,$$

Therefore

$$e^u = r.$$

By definition,

$$u = \ln r.$$

Since $e^w = z$,

$$e^u e^{iv} = re^{iv} = re^{i\theta}$$

it follows that:

$$v = \theta.$$

Thus

$$w = u + iv = \ln r + i\theta.$$

Therefore the rule of logarithm is preserved,

$$\ln z = \ln r\, e^{i\theta} = \ln r + i\theta. \tag{1.20}$$

Since θ is the polar angle, after it is increased by 2π in the z complex plane, it comes back to the same point and z will have the same value. However, the logarithm of z will not return to its original value. Its imaginary part will increase by $2\pi i$. If the argument of z in a particular interval of 2π is denoted as θ_0, then (1.20) can be written as

$$\ln z = \ln r + i(\theta_0 + 2\pi n), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

By specifying such an interval, we say that we have selected a particular branch of θ as the principal branch. The value corresponding to $n = 0$ is known as the principal value and is commonly denoted as $Ln\ z$, that is

$$Ln\ z = \ln r + i\theta_0.$$

The choice of the principal branch is somewhat arbitrary.

Figure 1.10 illustrates two possible branch selections. Figure 1.10a depicts the branch that selects the value of the argument of z from the interval $-\pi < \theta \leq \pi$. The values in this branch are most commonly used in complex algebra computer codes. The argument θ is inherently discontinuous, jumping by 2π as z crosses the negative x-axis. This line of discontinuities is known as the branch cut. The cut ends at the origin, which is known as the branch point.

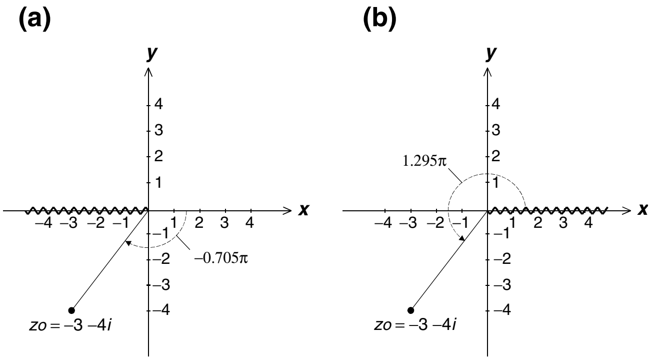


Fig. 1.10. Two possible branch selections. (**a**) Branch cut on the negative x-axis. The point $-3-4\mathrm{i}$ has argument -0.705π. (**b**) Branch cut on the positive x-axis. The point $-3-4\mathrm{i}$ has argument 1.295π

With the branch cut along the negative real axis, the principal value of the logarithm of $z_0 = -3-4\mathrm{i}$ is given by $\ln(|z_0|\,\mathrm{e}^{\mathrm{i}\theta})$ where $\theta = \tan^{-1}\frac{4}{3} - \pi$, thus the principal value is

$$\ln(-3-4\mathrm{i}) = \ln 5\mathrm{e}^{\mathrm{i}\theta} = \ln 5 \ + \mathrm{i}(\tan^{-1}\frac{4}{3} - \pi) = 1.609 - 0.705\pi\mathrm{i}.$$

However, if we select the interval $0 \leq \theta < 2\pi$ as the principal branch, then the branch cut is along the positive x-axis, as shown in Fig. 1.10b. In this case the principal value of the logarithm of z_0 is

$$\ln(-3-4\mathrm{i}) = \ln 5 \ + \mathrm{i}(\tan^{1}\frac{4}{3} + \pi) = 1.609 + 1.295\pi\mathrm{i}.$$

Unless otherwise specified, we shall use the interval $0 \leq \theta < 2\pi$ as the principal branch.

It can be easily checked that the familiar laws of logarithm which hold for real variables can be established for complex variables as well. For example,

$$\begin{aligned}
\ln z_1 z_2 &= \ln r_1\,\mathrm{e}^{\mathrm{i}\theta_1} r_2\,\mathrm{e}^{\mathrm{i}\theta_2} = \ln r_1 r_2 + \mathrm{i}\,(\theta_1 + \theta_2) \\
&= \ln r_1 + \ln r_2 + \mathrm{i}\theta_1 + \mathrm{i}\theta_2 \\
&= (\ln r_1 + \mathrm{i}\theta_1) + (\ln r_2 + \mathrm{i}\theta_2) = \ln z_1 + \ln z_2.
\end{aligned}$$

This relation is always true as long as infinitely many values of logarithms are taken into consideration. However, if only the principal values are taken, then the sum of the two principal values $\ln z_1 + \ln z_2$ may fall outside of the principal branch of $\ln(z_1 z_2)$.

Example 1.7.7. Find all values of $\ln 2$.

Solution 1.7.7. The real number 2 is also the complex number $2 + \mathrm{i}0$, and

$$2 + \mathrm{i}0 = 2\mathrm{e}^{\mathrm{i}n2\pi}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Thus

$$\begin{aligned} \ln 2 &= Ln\ 2 + n2\pi\mathrm{i} \\ &= 0.693 + n2\pi\mathrm{i}, \qquad n = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

Even positive real numbers now have infinitely many logarithms. Only one of them is real, corresponding to $n = 0$ principal value.

Example 1.7.8. Find all values of $\ln(-1)$.

Solution 1.7.8.

$$\ln(-1) = \ln \mathrm{e}^{\mathrm{i}(\pi \pm 2\pi n)} = \mathrm{i}(\pi + 2\pi n), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

The principal value is $\mathrm{i}\pi$ for $n = 0$.

Since $\ln a = x$ means $\mathrm{e}^x = a$, so long as the variable x is real, a is always positive. Thus, in the domain of real numbers, the logarithm of a negative number does not exist. Therefore the answer must come from the complex domain. The situation was still sufficiently confused in the 18th century that it was possible for so great a mathematician as D'Alembert (1717–1783) to think $\ln(-x) = \ln(x)$, so $\ln(-1) = \ln(1) = 0$. His reason was the following. Since $(-x)(-x) = x^2$, therefore $\ln[(-x)(-x)] = \ln x^2 = 2\ln x$. But $\ln[(-x)(-x)] = \ln(-x) + \ln(-x) = 2\ln(-x)$, so we get $\ln(-x) = \ln x$. This is incorrect, because it applies the rule of ordinary algebra to the domain of complex numbers. It was Euler who pointed out that $\ln(-1)$ must be equal to the complex number $\mathrm{i}\pi$, which is in accordance with his equation $\mathrm{e}^{\mathrm{i}\pi} = -1$.

Example 1.7.9. Find the principal value of $\ln(1 + \mathrm{i})$.

Solution 1.7.9. Since

$$1 + \mathrm{i} = \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4},$$

$$\ln(1 + \mathrm{i}) = \ln\sqrt{2} + \frac{\pi}{4}\mathrm{i} = 0.3466 + 0.7854\mathrm{i}.$$

We are now in a position to consider the general power of a complex number. First let us see how to find i^{i}. Since

$$\mathrm{i} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi n\right)},$$

$$\mathrm{i}^{\mathrm{i}} = \left[\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi n\right)}\right]^{\mathrm{i}} = \mathrm{e}^{-\left(\frac{\pi}{2}+2\pi n\right)}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

We get infinitely many values – all of them real. In a literal sense, Euler showed that imaginary power of an imaginary number can be real.

In general, since $a = \mathrm{e}^{\ln a}$, so

$$a^b = \left(\mathrm{e}^{\ln a}\right)^b = \mathrm{e}^{b \ln a}.$$

In this formula, both a and b can be complex numbers. For example, to find $(1+\mathrm{i})^{1-\mathrm{i}}$, first we write

$$(1+i)^{1-i} = \left[\mathrm{e}^{\ln(1+i)}\right]^{1-i} = \mathrm{e}^{(1-i)\ln(1+i)}.$$

Since

$$\ln(1+\mathrm{i}) = \ln \sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+2\pi n\right)} = \ln \sqrt{2} + \mathrm{i}\left(\frac{\pi}{4} + 2\pi n\right), \qquad n = 0, \pm 1, \pm 2, \ldots,$$

now

$$\begin{aligned}(1+\mathrm{i})^{1-\mathrm{i}} &= \mathrm{e}^{\left(\ln\sqrt{2}+\mathrm{i}\left(\frac{\pi}{4}+2\pi n\right)-\mathrm{i}\ln\sqrt{2}+\left(\frac{\pi}{4}+2\pi n\right)\right)} \\ &= \mathrm{e}^{\ln\sqrt{2}+\frac{\pi}{4}+2\pi n}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+2\pi n-\ln\sqrt{2}\right)}.\end{aligned}$$

Using

$$\mathrm{e}^{\mathrm{i}2\pi n} = 1, \qquad \mathrm{e}^{\ln\sqrt{2}} = \sqrt{2},$$

we have

$$(1+\mathrm{i})^{1-\mathrm{i}} = \sqrt{2}\mathrm{e}^{\frac{\pi}{4}+2\pi n}\left[\cos\left(\frac{\pi}{4}-\ln\sqrt{2}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}-\ln\sqrt{2}\right)\right].$$

Using a calculator, this expression is found to be

$$(1+\mathrm{i})^{1-\mathrm{i}} = \mathrm{e}^{2\pi n}\left(2.808 + 1.318\mathrm{i}\right), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Example 1.7.10. Find all values of $\mathrm{i}^{1/2}$.

Solution 1.7.10.

$$\mathrm{i}^{1/2} = \left[\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi n\right)}\right]^{1/2} = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}n\pi}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Since $\mathrm{e}^{\mathrm{i}n\pi} = 1$ for n even, and $\mathrm{e}^{\mathrm{i}n\pi} = -1$ for n odd, thus

$$\mathrm{i}^{1/2} = \pm\mathrm{e}^{\mathrm{i}\pi/4} = \pm\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = \pm\frac{\sqrt{2}}{2}\left(1+\mathrm{i}\right).$$

Notice that although n can be any of the infinitely many integers, we find only two values for $\mathrm{i}^{1/2}$ as we should, for it is the square root of i.

Example 1.7.11. Find the principal value of 2^{i}.

Solution 1.7.11.

$$\begin{aligned} 2^{\mathrm{i}} &= \left[\mathrm{e}^{\ln 2}\right]^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\ln 2} = \cos(\ln 2) + \mathrm{i}\sin(\ln 2) \\ &= 0.769 + 0.639\mathrm{i}. \end{aligned}$$

Example 1.7.12. Find the principal value of $(1+\mathrm{i})^{2-\mathrm{i}}$.

Solution 1.7.12.

$$(1+\mathrm{i})^{2-\mathrm{i}} = \exp\left[(2-\mathrm{i})\ln(1+\mathrm{i})\right].$$

The principal value of $\ln(1+\mathrm{i})$ is

$$\ln(1+\mathrm{i}) = \ln\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4} = \ln\sqrt{2} + \mathrm{i}\frac{\pi}{4}.$$

Therefore

$$\begin{aligned} (1+\mathrm{i})^{2-\mathrm{i}} &= \exp\left[(2-\mathrm{i})\left(\ln\sqrt{2} + \mathrm{i}\frac{\pi}{4}\right)\right] \\ &= \exp\left(2\ln\sqrt{2} + \frac{\pi}{4}\right)\exp\left[\mathrm{i}\left(\frac{\pi}{2} - \ln\sqrt{2}\right)\right] \\ &= 2\mathrm{e}^{\pi/4}\left[\cos\left(\frac{\pi}{2} - \ln\sqrt{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2} - \ln\sqrt{2}\right)\right] \\ &= 4.3866\left(\sin 0.3466 + \mathrm{i}\cos 0.3466\right) = 1.490 + 4.126\mathrm{i}. \end{aligned}$$

1.7.4 Inverse Trigonometric and Hyperbolic Functions

Starting from their definitions, we can work out sensible expressions for the inverse of trigonometric and inverse hyperbolic functions. For example, to find

$$w = \sin^{-1} z,$$

we write this as

$$z = \sin w = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}\right).$$

Multiplying $\mathrm{e}^{\mathrm{i}w}$, we have

$$z\mathrm{e}^{\mathrm{i}w} = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}2w} - 1\right).$$

Rearranging, we get a quadratic equation in $\mathrm{e}^{\mathrm{i}w}$,

$$\left(\mathrm{e}^{\mathrm{i}w}\right)^2 - 2\mathrm{i}z\mathrm{e}^{\mathrm{i}w} - 1 = 0.$$

The solution of this equation is

$$\mathrm{e}^{\mathrm{i}w} = \frac{1}{2}\left(2\mathrm{i}z \pm \sqrt{-4z^2+4}\right) = \mathrm{i}z \pm \left(1-z^2\right)^{1/2}.$$

Taking logarithm of both sides

$$\mathrm{i}w = \ln\left[\mathrm{i}z \pm \left(1-z^2\right)^{1/2}\right].$$

Therefore

$$w = \sin^{-1} z = -\mathrm{i}\ln\left[\mathrm{i}z \pm \left(1-z^2\right)^{1/2}\right].$$

Because of logarithm, this expression is multivalued. Even in the principal branch, $\sin^{-1} z$ has two values for $z \neq 1$ because of the square roots.

Similarly, we can show

$$\cos^{-1} z = -\mathrm{i}\ln\left[z \pm \left(z^2-1\right)^{1/2}\right],$$
$$\tan^{-1} z = \frac{\mathrm{i}}{2}\ln\frac{\mathrm{i}+z}{\mathrm{i}-z},$$

$$\sinh^{-1} z = \ln\left[z \pm \left(1+z^2\right)^{1/2}\right],$$
$$\cosh^{-1} z = \ln\left[z \pm \left(z^2-1\right)^{1/2}\right],$$
$$\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}.$$

Example 1.7.13. Evaluate $\cos^{-1} 2$.

Solution 1.7.13. Let $w = \cos^{-1} 2$, so $\cos w = 2$. It follows:

$$\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}w} + \mathrm{e}^{-\mathrm{i}w}\right) = 2$$

Multiplying $\mathrm{e}^{\mathrm{i}w}$, we have a quadratic equation in $\mathrm{e}^{\mathrm{i}w}$

$$(\mathrm{e}^{\mathrm{i}w})^2 + 1 = 4\mathrm{e}^{\mathrm{i}w}.$$

Solving for $\mathrm{e}^{\mathrm{i}w}$

$$\mathrm{e}^{\mathrm{i}w} = \frac{1}{2}(4 \pm \sqrt{16-4}) = 2 \pm \sqrt{3}.$$

Thus

$$\mathrm{i}w = \ln\left(2 \pm \sqrt{3}\right).$$

Therefore

$$\cos^{-1} 2 = w = -\mathrm{i}\ln\left(2 \pm \sqrt{3}\right).$$

Now

$$\ln(2+\sqrt{3}) = 1.317, \qquad \ln(2-\sqrt{3}) = -1.317.$$

Note only in this particular case, $-\ln(2+\sqrt{3}) = \ln(2-\sqrt{3})$, since

$$-\ln(2+\sqrt{3}) = \ln(2+\sqrt{3})^{-1} = \ln\frac{1}{2+\sqrt{3}} = \ln\frac{2-\sqrt{3}}{2^2-(\sqrt{3})^2} = \ln(2-\sqrt{3}).$$

Thus the principal values of $\ln\left(2 \pm \sqrt{3}\right) = \pm 1.317$. Therefore

$$\cos^{-1} 2 = \mp 1.317\mathrm{i} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots.$$

In real variable domain, the maximum value of cosine is one. Therefore we expect $\cos^{-1} 2$ to be complex numbers. Also note that $\pm$ solutions may be expected since $\cos(-z) = \cos(z)$.

Example 1.7.14. Show that

$$\tan^{-1} z = \frac{\mathrm{i}}{2}[\ln(\mathrm{i}+z) - \ln(\mathrm{i}-z)].$$

Solution 1.7.14. Let $w = \tan^{-1} z$, so

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}}{\mathrm{i}\left(\mathrm{e}^{\mathrm{i}w} + \mathrm{e}^{-\mathrm{i}w}\right)},$$

$$\mathrm{i}z\left(\mathrm{e}^{\mathrm{i}w} + \mathrm{e}^{-\mathrm{i}w}\right) = \mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w},$$

$$(\mathrm{i}z - 1)\,\mathrm{e}^{\mathrm{i}w} + (\mathrm{i}z + 1)\,\mathrm{e}^{-\mathrm{i}w} = 0.$$

Multiplying $\mathrm{e}^{\mathrm{i}w}$ and rearranging, we have

$$\mathrm{e}^{\mathrm{i}2w} = \frac{1+\mathrm{i}z}{1-\mathrm{i}z}.$$

Taking logarithm on both sides

$$\mathrm{i}\, 2w = \ln \frac{1 + \mathrm{i}z}{1 - \mathrm{i}z} = \ln \frac{\mathrm{i} - z}{\mathrm{i} + z}.$$

Thus

$$w = \frac{1}{2\mathrm{i}} \ln \frac{\mathrm{i} - z}{\mathrm{i} + z} = -\frac{\mathrm{i}}{2} \ln \frac{\mathrm{i} - z}{\mathrm{i} + z} = \frac{\mathrm{i}}{2} \ln \frac{\mathrm{i} + z}{\mathrm{i} - z},$$

$$\tan^{-1} z = w = \frac{\mathrm{i}}{2}[\ln(\mathrm{i} + z) - \ln(\mathrm{i} - z)].$$

Exercises

1. Approximate $\sqrt{2}$ as 1.414 and use the table of successive square root of 10 to compute $10^{\sqrt{2}}$.

   Ans. 25.94

2. Use the table of successive square root of 10 to compute $\log 2$.

   Ans. 0.3010

3. How long will it take for a sum of money to double if invested at 20% interest rate compounded annually? (This question was posted in a clay tablet dated 1700 BC now at Louvre.)
   Hint: Solve $(1.2)^x = 2$.

   Ans. 3.8 years, or 3 years 8 months and 18 days.

4. Suppose the annual interest rate is fixed at 5%. Banks are competing by offering compound interests with increasing number of conversions, monthly, daily, hourly, and so on. With a principal of \$100, what is the maximum amount of money one can get after 1 year?

   Ans. $100\,\mathrm{e}^{0.05} = 105.13$

5. Simplify (express it in the form of $a + \mathrm{i}b$)

$$\frac{\cos 2\alpha + \mathrm{i} \sin 2\alpha}{\cos \alpha + \mathrm{i} \sin \alpha}.$$

   Ans. $\cos \alpha + \mathrm{i} \sin \alpha$.

6. Simplify (express it in the form of $a + \mathrm{i}b$)

$$\frac{(\cos \theta - \mathrm{i} \sin \theta)^2}{(\cos \theta + \mathrm{i} \sin \theta)^3}.$$

   Ans. $\cos 5\theta - \mathrm{i} \sin 5\theta$.

7. Find the roots of

$$x^4 + 1 = 0.$$

Ans. $\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}.$

8. Find all the distinct fourth roots of $8 - \mathrm{i}8\sqrt{3}$.
Ans. $2\left(\cos\frac{5\pi}{12} + \mathrm{i}\sin\frac{5\pi}{12}\right), 2\left(\cos\frac{11\pi}{12} + \mathrm{i}\sin\frac{11\pi}{12}\right),$
$2\left(\cos\frac{17\pi}{12} + \mathrm{i}\sin\frac{17\pi}{12}\right), 2\left(\cos\frac{23\pi}{12} + \mathrm{i}\sin\frac{23\pi}{12}\right).$

9. Find all the values of the following in the form of $a + \mathrm{i}b$.

$$\text{(a) } \mathrm{i}^{2/3}, \quad \text{(b) } (-1)^{1/3}, \quad \text{(c) } (3 + 4\mathrm{i})^4.$$

Ans. (a) $-1,\ (1 \pm \mathrm{i}\sqrt{3})/2$, (b) $-1,\ (1 \pm \mathrm{i}\sqrt{3})/2$, (c) $-527 - 336\mathrm{i}$.

10. Use complex numbers to show

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

11. Use complex numbers to show

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1),$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

12. Show that

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\frac{1}{2}\theta},$$

$$\sum_{k=0}^{n} \sin k\theta = \frac{1}{2}\cot\frac{1}{2}\theta - \frac{\cos\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\frac{1}{2}\theta}.$$

13. Find the location of the center and the radius of the following circle:

$$\left|\frac{z - 1}{z + 1}\right| = 3.$$

Ans. $(-\frac{5}{4},\ 0)\quad r = \frac{3}{4}$.

14. Use complex numbers to show that the diagonals of a parallelogram bisect each other.

15. Use complex numbers to show that the line segment joining the two midpoints of two sides of any triangle is parallel to the third side and half its length.

16. Use complex numbers to prove that medians of a triangle intersect at a point two-thirds of the way from any vertex to the midpoint of the opposite side.

17. Let ABC be an isosceles triangle such that $AB = AC$. Use complex numbers to show that the line from A to the midpoint of BC is perpendicular to BC.

18. Express the principal value of the following in the form of $a + \mathrm{i}b$:
$$(a)\ \exp\left(\frac{\mathrm{i}\pi}{4} + \frac{\ln 2}{2}\right), \quad (b)\ \cos(\pi - 2\mathrm{i}\ln 3), \quad (c)\ \ln(-\mathrm{i}).$$
Ans. (a) $1 + \mathrm{i}$, (b) $-\frac{41}{9}$, (c) $-\mathrm{i}\frac{\pi}{2}$ or $\mathrm{i}\frac{3\pi}{2}$.

19. Express the principal value of the following in the form of $a + \mathrm{i}b$:
$$\text{(a) } \mathrm{i}^{3+\mathrm{i}}, \ \text{(b) } (2\mathrm{i})^{1+\mathrm{i}}, \text{(c) } \left(\frac{1 + \mathrm{i}\sqrt{3}}{2}\right)^{\mathrm{i}}.$$
Ans. (a) $-0.20788\mathrm{i}$, (b) $-0.2657 + 0.3189\mathrm{i}$, (c) 0.35092.

20. Find all the values of the following expressions:
$$\text{(a) } \sin\left(\mathrm{i}\ \ln\frac{1-\mathrm{i}}{1+\mathrm{i}}\right), (b)\ \tan^{-1}(2\mathrm{i}), (c)\ \cosh^{-1}\left(\frac{1}{2}\right)$$
Ans. (a) 1, (b) $\frac{1+2n}{2}\pi + \mathrm{i}\frac{1}{2}\ln 3$, (c) $\mathrm{i}\left(\pm\frac{\pi}{3} + 2n\pi\right)$.

21. With $z = x + \mathrm{i}y$, verify the following
$$\begin{aligned}
\sin z &= \sin x \cosh y + \mathrm{i}\cos x \sinh y, \\
\cos z &= \cos x \cosh y - \mathrm{i}\sin x \sinh y, \\
\sinh z &= \sinh x \cos y + \mathrm{i}\cosh x \sin y, \\
\cosh z &= \cosh x \cos y + \mathrm{i}\sinh x \sin y.
\end{aligned}$$

22. Show that
$$\begin{aligned}
\sin 2z &= 2\sin z \cos z, \\
\cos 2z &= \cos^2 z - \sin^2 z, \\
\cosh^2 z - \sinh^2 z &= 1.
\end{aligned}$$

23. Show that
$$\begin{aligned}
\cos^{-1} z &= -\mathrm{i}\ \ln\left[z + \left(z^2 - 1\right)^{1/2}\right], \\
\sinh^{-1} z &= \ln\left[z + \left(1 + z^2\right)^{1/2}\right].
\end{aligned}$$

2

Complex Functions

Complex numbers were first used to simplify calculations. In the course of time, it became clear that the theory of complex functions is a very effective tool in engineering and sciences. Often the most elegant solutions of important problems in heat conduction, elasticity, electrostatics, and hydrodynamics are produced by complex function methods. In modern physics, complex variables have even become an intrinsic part of the physical theory. For example, it is a fundamental postulate in quantum mechanics that wave functions reside in a complex vector space.

In engineering and sciences the ultimate test is in the laboratory. When you make a measurement, the result you get is, of course, a real number. But the theoretical formulation of the problem often leads us into the realm of complex numbers. It is almost a miracle that, if the theory is correct, further mathematical analysis with complex functions will always lead us to an answer that is real. Therefore the theory of complex functions is an essential tool in modern sciences.

Complex functions to which the concepts and structure of calculus can be applied are called analytic functions. It is the analytic functions that dominate complex analysis. Many interesting properties and applications of analytic functions are studied in this chapter.

2.1 Analytic Functions

The theory of analytic functions is an extension of the differential and integral calculus to realms of complex variables. However, the notion of a derivative of a complex function is far more subtle than that of a real function. This is because of the intrinsically two-dimensional nature of the complex numbers. The success made in analyzing this question by Cauchy and Riemann left a deep imprint on the whole of mathematics. It also had a far reaching consequences in several branches of mathematical physics.

2.1.1 Complex Function as Mapping Operation

From the complex variable $z = x + \mathrm{i}y$, one can construct complex functions $f(z)$. Formally we can define functions of complex variables in exactly the same way as functions of real variables are defined, except allowing the constants and variables to assume complex values.

Let $w = f(z)$ denote some functional relationship connecting w and z. These functions may then be resolved into real and imaginary parts

$$w = f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$$

in which both $u(x, y)$ and $v(x, y)$ are real functions. For example, if

$$w = f(z) = z^2,$$

then

$$w = (x + \mathrm{i}y)^2 = (x^2 - y^2) + \mathrm{i}2xy.$$

So the real and imaginary parts of $w\,(u, v)$ are, respectively,

$$u(x, y) = (x^2 - y^2), \tag{2.1}$$

$$v(x, y) = 2xy. \tag{2.2}$$

Since two dimensions are needed to specify the independent variable $z(x, y)$ and another two dimensions to specify the dependent variable $w\,(u, v)$, a complex function cannot be represented by a single two- or three-dimensional plot. The functional relationship $w = f(z)$ is perhaps best pictured as a mapping, or a transformation, operation. A set of points (x, y), in the z-plane $(z = x + \mathrm{i}y)$ are mapped into another set of points (u, v), in the w-plane $(w = u + \mathrm{i}v)$. If we allow the variable x and y to trace some curve in the z-plane, this will force the variable u and v to trace an image curve in the w-plane.

In the above example, if the point (x, y) in the z-plane moves along the hyperbola $x^2 - y^2 = c$ (where c is a constant), the image point given by (2.1) will move along the curve $u = c$, that is a vertical line in the w-plane. Similarly, if the point moves along the hyperbola $2xy = k$, the image point given by (2.2) will trace the horizontal line $v = k$ in the w-plane. The hyperbolas $x^2 - y^2 = c$ and $2xy = k$ form two families of curves in the z-plane, each curve corresponding to a given value of the constant c or k. Their image curves form a rectangular grid of horizontal and vertical lines in the w-plane, as shown in Fig. 2.1.

2.1.2 Differentiation of a Complex Function

To discuss the differentiation of a complex function $f\,(z)$ at certain point z_0, the function must be defined in some neighborhood of the point z_0. By the neighborhood we mean the set of all points in a sufficiently small circular

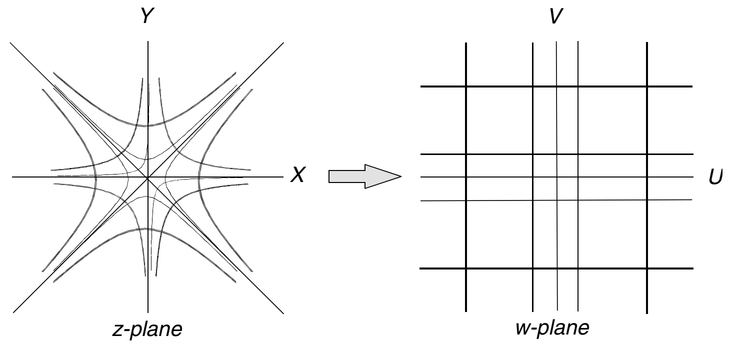


Fig. 2.1. The function $w = z^2$ maps hyperbolas in the z-plane onto horizontal and vertical lines in the w-plane

region with center at z_0. If $z_0 = x_0 + \mathrm{i}y_0$ and $z = z_0 + \Delta z$ are two nearby points in the z-plane with $\Delta z = \Delta x + \mathrm{i}\,\Delta y$, the corresponding image points in the w-plane are $w_0 = u_0 + \mathrm{i}v_0$ and $w = w_0 + \Delta w$, where $w_0 = f(z_0)$ and $w = f(z) = f(z_0 + \Delta z)$. The change Δw caused by the increment Δz in z_0 is

$$\Delta w = f(z_0 + \Delta z) - f(z_0)\,.$$

These functional relationships are shown in Fig. 2.2.

Now we define the derivative $f'(z) = \frac{\mathrm{d}w}{\mathrm{d}z}$ by the usual formula

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2.3}$$

It is most important to note that in this formula $z = z_0 + \Delta z$ can assume any position in the neighborhood of z_0 and Δz can approach zero along any of the infinitely many paths joining z with z_0. Hence if the derivative is to have a unique value, we must demand that the limit be independent of the way in which Δz is made to approach zero. This restriction greatly narrows down the class of complex functions that possess derivatives.

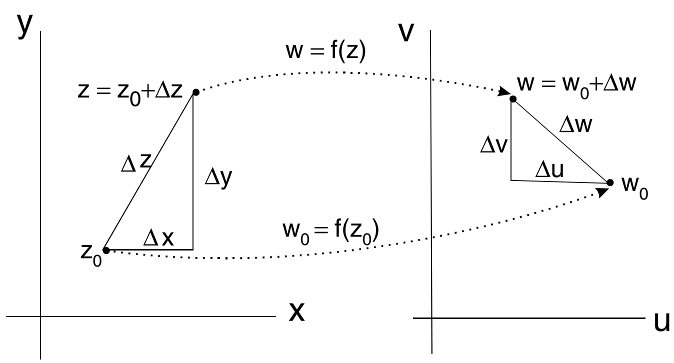


Fig. 2.2. The neighborhood of z_0 in the z-plane is mapped onto the neighborhood of w_0 in the w-plane by the function $w = f(z)$

For example, if $f(z) = |z|^2$, then $w = zz^*$, and

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(z^* + \Delta z^*) - zz^*}{\Delta z} \\ &= z^* + \Delta z^* + \frac{z\Delta z^*}{\Delta z} = x - \mathrm{i}y + \Delta x - \mathrm{i}\Delta y + (x+\mathrm{i}y)\frac{\Delta x - \mathrm{i}\Delta y}{\Delta x + \mathrm{i}\Delta y}.\end{aligned}$$

For the derivative $f'(z)$ to exist, the limit of this quotient must be the same no matter how Δz approaches zero. Since $\Delta z = \Delta x + \mathrm{i}\Delta y$, $\Delta z \to 0$ means, of course, both $\Delta x \to 0$ and $\Delta y \to 0$. However, the way they go to zero may make a difference. If we let Δz approach zero along path I in Fig. 2.3, so that first $\Delta y \to 0$ and then $\Delta x \to 0$, we get

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta x \to 0} \left\{ \lim_{\Delta y \to 0} \left[x - \mathrm{i}y + \Delta x - \mathrm{i}\Delta y + (x+\mathrm{i}y)\frac{\Delta x - \mathrm{i}\Delta y}{\Delta x + \mathrm{i}\Delta y} \right] \right\} \\ &= 2x.\end{aligned}$$

But if we take path II and first allow $\Delta x \to 0$ and then $\Delta y \to 0$, we obtain

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta y \to 0} \left\{ \lim_{\Delta x \to 0} \left[x - \mathrm{i}y + \Delta x - \mathrm{i}\Delta y + (x+\mathrm{i}y)\frac{\Delta x - \mathrm{i}\Delta y}{\Delta x + \mathrm{i}\Delta y} \right] \right\} \\ &= -2\mathrm{i}y.\end{aligned}$$

These limits are different, and hence $w = |z|^2$ has no derivative except possibly at $z = 0$.

On the other hand, if we consider $w = z^2$, then

$$w + \Delta w = (z + \Delta z)^2 = z^2 + 2z\,\Delta z + (\Delta z)^2,$$

so that

$$\frac{\Delta w}{\Delta z} = \frac{2z\,\Delta z + (\Delta z)^2}{\Delta z} = 2z + \Delta z.$$

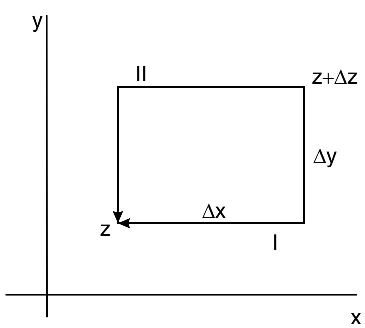


Fig. 2.3. To be differentiable at z, the same limit must be obtained no matter which path Δz is taken to approach zero

The limit of this quotient as $\Delta z \to 0$ is invariably $2z$, whatever may be the path along which Δz approaches zero. Therefore the derivative exists everywhere and

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

It is clear that not every combination of $u(x,y) + \mathrm{i}v(x,y)$ can be differentiated with respect to z. If a complex function $f(z)$ whose derivative $f'(z)$ exists at z_0 and at every point in the neighborhood of z_0, then the function is said to be analytic at z_0. An analytic function is a function that is analytic in some region (domain) of the complex plane. A function that is analytic in the whole complex plane is called an entire function. A point at which an analytic function ceases to have a derivative is called a singular point.

2.1.3 Cauchy–Riemann Conditions

We will now investigate the conditions that a complex function must satisfy in order to be differentiable.

It follows from the definition:

$$f(z) = f(x + \mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y),$$

that

$$\begin{aligned} f(z + \Delta z) &= f((x + \Delta x) + \mathrm{i}(y + \Delta y)) \\ &= u(x + \Delta x, y + \Delta y) + \mathrm{i}v(x + \Delta x, y + \Delta y). \end{aligned}$$

Since $w = f(z)$ and $w + \Delta w = f(z + \Delta z)$, so

$$\Delta w = f(z + \Delta z) - f(z) = \Delta u + \mathrm{i}\,\Delta v,$$

where

$$\begin{aligned} \Delta u &= u(x + \Delta x, y + \Delta y) - u(x,y), \\ \Delta v &= v(x + \Delta x, y + \Delta y) - v(x,y). \end{aligned}$$

We can add $0 = -u(x, y + \Delta y) + u(x, y + \Delta y)$ to Δu without changing its value

$$\begin{aligned} \Delta u &= u(x + \Delta x, y + \Delta y) - u(x,y) \\ &= u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) + u(x, y + \Delta y) - u(x,y). \end{aligned}$$

Recall the definition of partial derivative

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) \right] = \frac{\partial u}{\partial x}.$$

In this expression only x variable is increased by Δx and y variable remains the same. If it is implicitly understood that the symbol Δx carries the meaning that it is approaching zero as a limit, then we can move it to the right-hand side

$$u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) = \frac{\partial u}{\partial x}\Delta x.$$

Similarly, in the following expression only y variable is increased by Δy, so:

$$u(x, y+\Delta y) - u(x, y) = \frac{\partial u}{\partial y}\Delta y.$$

Therefore

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y. \tag{2.4}$$

Likewise,

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y.$$

Hence the derivative given by (2.3) can be written as

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z\to 0}\frac{\Delta u + \mathrm{i}\Delta v}{\Delta x + \mathrm{i}\Delta y} = \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})\Delta x + (\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})\Delta y}{\Delta x + \mathrm{i}\,\Delta y}.$$

Dividing both the numerator and denominator by Δx, we have

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}z} &= \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}) + (\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})\frac{\Delta y}{\Delta x}}{1 + \mathrm{i}\frac{\Delta y}{\Delta x}} \\ &= \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})}{1 + \mathrm{i}\frac{\Delta y}{\Delta x}}\left[1 + \frac{(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})}{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})}\frac{\Delta y}{\Delta x}\right].\end{aligned}$$

There are infinitely many paths that Δz can approach zero, each path is characterized by its slope $\frac{\Delta y}{\Delta x}$ as shown in Fig. 2.4. For all these paths to give the same limit, $\frac{\Delta y}{\Delta x}$ must be eliminated from this expression. This will be the case if and only if

$$\frac{(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})}{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})} = \mathrm{i}, \tag{2.5}$$

since then the expression becomes

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})}{1 + \mathrm{i}\frac{\Delta y}{\Delta x}}\left[1 + \mathrm{i}\frac{\Delta y}{\Delta x}\right] = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}, \tag{2.6}$$

which is independent of $\frac{\Delta y}{\Delta x}$.

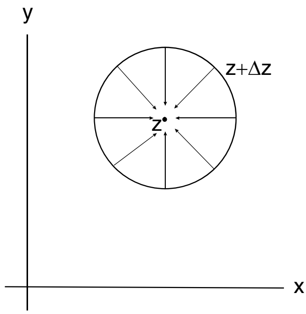


Fig. 2.4. Infinitely many paths Δz can approach zero, each characterized by its slope

From (2.5), we have

$$\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y} = \mathrm{i}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}.$$

Equating the real and imaginary parts, we arrive at the following pair of equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These two equations are extremely important and are known as Cauchy–Riemann equations.

With the Cauchy–Riemann equations, the derivative shown in (2.6) can be written as

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial u}{\mathrm{i}\partial y} + \mathrm{i}\frac{\partial v}{\mathrm{i}\partial y}. \tag{2.7}$$

The expression in (2.6) is the derivative with Δz approaching zero along the real x-axis and the expression in (2.7) is the derivative with Δz approaching zero along the imaginary y-axis. For an analytic function, they must be the same.

Thus if $u(x,y)$, $v(x,y)$ are continuous and satisfy the Cauchy–Riemann equations in some region of the complex plane, then $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is an analytic function in that region. In other words, Cauchy–Riemann equations are necessary and sufficient conditions for the function to be differentiable.

2.1.4 Cauchy–Riemann Equations in Polar Coordinates

Often the function $f(z)$ is expressed in polar coordinates, so it is convenient to express the Cauchy–Riemann equations in polar form.

Since $x = r\cos\theta$ and $y = r\sin\theta$, so by chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta.$$

Similarly,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta,$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta.$$

With the Cauchy–Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we have

$$\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta,$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}r\sin\theta + \frac{\partial u}{\partial x}r\cos\theta.$$

Thus

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\ \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

are the Cauchy–Riemann equations in the polar form.

It is instructive to derive these equations from the definition of the derivative

$$f'(z) = \lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z}.$$

In the polar coordinates, $z = r\,\mathrm{e}^{\mathrm{i}\theta}$,

$$\Delta w = \Delta u\,(r,\theta) + \mathrm{i}\Delta v\,(r,\theta)\,,$$
$$\Delta z = (r+\Delta r)\,\mathrm{e}^{\mathrm{i}(\theta+\Delta\theta)} - r\,\mathrm{e}^{\mathrm{i}\theta}.$$

For $\Delta z \to 0$, we can first let $\Delta\theta \to 0$ and obtain

$$\Delta z = (r+\Delta r)\,\mathrm{e}^{\mathrm{i}\theta} - r\,\mathrm{e}^{\mathrm{i}\theta} = \Delta r\,\mathrm{e}^{\mathrm{i}\theta},$$

and then let $\Delta r \to 0$, so

$$f'(z) = \lim_{\Delta r \to 0}\frac{\Delta u\,(r,\theta) + \mathrm{i}\Delta v\,(r,\theta)}{\Delta r\,\mathrm{e}^{\mathrm{i}\theta}} = \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right).$$

But if we let $\Delta r \to 0$ first, we get

$$\Delta z = r\,\mathrm{e}^{\mathrm{i}(\theta+\Delta\theta)} - r\,\mathrm{e}^{\mathrm{i}\theta}.$$

Since

$$\mathrm{e}^{\mathrm{i}(\theta+\Delta\theta)} - \mathrm{e}^{\mathrm{i}\theta} = \frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{d}\theta}\Delta\theta = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\Delta\theta,$$

so Δz can be written as

$$\Delta z = r\,\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\Delta\theta,$$

and when we take the limit $\Delta\theta \to 0$, the derivative becomes

$$f'(z) = \lim_{\Delta\theta\to 0} \frac{\Delta u\,(r,\theta) + \mathrm{i}\Delta v\,(r,\theta)}{r\,\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\Delta\theta} = \frac{1}{\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial\theta} + \mathrm{i}\frac{\partial v}{\partial\theta}\right).$$

For an analytic function, the two expressions of derivative must be the same,

$$\frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right) = \frac{1}{\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial\theta} + \mathrm{i}\frac{\partial v}{\partial\theta}\right).$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta},$$

which is what we obtained by direct transformation.

Furthermore, the derivative is given by either of the equivalent expressions

$$\begin{aligned} f'(z) &= \mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right) \\ &= \frac{1}{\mathrm{i}\,r}\mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial\theta} + \mathrm{i}\frac{\partial v}{\partial\theta}\right). \end{aligned}$$

2.1.5 Analytic Function as a Function of z Alone

In any analytic function $w = u(x,y) + \mathrm{i}v(x,y)$, the variables x, y can be replaced by their equivalents in terms of z, z^*:

$$x = \frac{1}{2}\left(z + z^*\right) \quad \text{and} \quad y = \frac{1}{2\mathrm{i}}\left(z - z^*\right),$$

since the complex variable $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$. Thus an analytic function can be regarded formally as a function of z and z^*. To show that w depends only on z and does not involve z^*, it is sufficient to compute $\frac{\partial w}{\partial z^*}$ and verify that it is identically zero. Now by chain rule

$$\begin{aligned}\frac{\partial w}{\partial z^*} &= \frac{\partial(u+\mathrm{i}v)}{\partial z^*} = \frac{\partial u}{\partial z^*} + \mathrm{i}\frac{\partial v}{\partial z^*} \\ &= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z^*}\right) + \mathrm{i}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z^*}\right).\end{aligned}$$

Since, from the equations expressing x and y in terms of z and z^*, we have

$$\frac{\partial x}{\partial z^*} = \frac{1}{2} \quad \text{and} \quad \frac{\partial y}{\partial z^*} = \frac{\mathrm{i}}{2},$$

we can write

$$\begin{aligned}\frac{\partial w}{\partial z^*} &= \left(\frac{1}{2}\frac{\partial u}{\partial x} + \frac{\mathrm{i}}{2}\frac{\partial u}{\partial y}\right) + \mathrm{i}\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{\mathrm{i}}{2}\frac{\partial v}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).\end{aligned}$$

Since w, by hypothesis, is an analytic function, u and v satisfy the Cauchy–Riemann conditions, therefore each of the quantities in parentheses in the last expression vanishes. Thus

$$\frac{\partial w}{\partial z^*} = 0. \tag{2.8}$$

Hence, w is independent of z^*, that is, it depends on x and y only through the combination $x + \mathrm{i}y$.

Therefore if w is an analytic function, then it can be written as

$$w = f(z),$$

and its derivative is defined as

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

This definition is formally identical with that for the derivatives of a function of a real variable. Since the general theory of limits is phrased in terms of absolute values, so if it is valid for real variables, it will also be valid for complex variables. Hence formulas in real variables will have counterparts in complex variables. For example, formulas such as

$$\frac{\mathrm{d}\,(w_1 \pm w_2)}{\mathrm{d}z} = \frac{\mathrm{d}w_1}{\mathrm{d}z} \pm \frac{\mathrm{d}w_2}{\mathrm{d}z},$$

$$\frac{\mathrm{d}\,(w_1 w_2)}{\mathrm{d}z} = w_1\frac{\mathrm{d}w_2}{\mathrm{d}z} + w_2\frac{\mathrm{d}w_1}{\mathrm{d}z},$$

$$\frac{\mathrm{d}\,(w_1/w_2)}{\mathrm{d}z} = \frac{w_2(\mathrm{d}w_1/\mathrm{d}z) - w_1(\mathrm{d}w_2/\mathrm{d}z)}{w_2^2}, \qquad w_2 \neq 0,$$

$$\frac{\mathrm{d}\left(w^n\right)}{\mathrm{d}z} = nw^{n-1}\frac{\mathrm{d}w}{\mathrm{d}z}$$

are all valid as long as w_1, w_2, and w are analytic functions.

Specifically, any polynomial in z

$$w\left(z\right) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

is analytic in the whole complex plane and therefore is an entire function. Its derivative is

$$w'\left(z\right) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1.$$

Consequently any rational function of z (a polynomial over another polynomial) is analytic at every point for which its denominator is not zero. At the zeros of the denominator, the function blows up and is not differentiable. Therefore the zeros of the denominator are the singular points of the function.

In fact we can take (2.8) as an alternative statement of the differentiability condition. Thus, the elementary functions defined in the previous chapter are all analytic functions, (some with singular points), since they are functions of z alone. It can be easily shown that they satisfy the Cauchy–Riemann conditions.

Example 2.1.1. Show that the real part u and the imaginary part v of $w = z^2$ satisfy the Cauchy–Riemann equations. Find the derivative of w through the partial derivatives of u and v.

Solution 2.1.1. Since

$$w = z^2 = (x + \mathrm{i}y)^2 = \left(x^2 - y^2\right) + \mathrm{i}2xy,$$

so the real and imaginary parts are

$$u\left(x, y\right) = x^2 - y^2, \qquad v\left(x, y\right) = 2xy.$$

Therefore

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Thus the Cauch–Riemann equations are satisfied. It is differentiable and

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = 2x + \mathrm{i}2y = 2z$$

which is what we found before regarding z as a single variable.

Example 2.1.2. Show that the real part u and the imaginary part v of $f(z) = \mathrm{e}^z$ satisfy the Cauchy–Riemann equations. Find the derivative of $f(z)$ through the partial derivatives of u and v.

Solution 2.1.2. Since

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)$$

the real and imaginary parts are, respectively,

$$u = \mathrm{e}^x \cos y \quad \text{and} \quad v = \mathrm{e}^x \sin y.$$

It follows that:

$$\frac{\partial u}{\partial x} = \mathrm{e}^x \cos y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\mathrm{e}^x \sin y = -\frac{\partial v}{\partial x}.$$

So the Cauchy–Riemann equations are satisfied, and

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \mathrm{e}^x \cos y + \mathrm{i}\mathrm{e}^x \sin y = \mathrm{e}^z,$$

which is what we expect by regarding z as a single variable.

Example 2.1.3. Show that the real part u and the imaginary part v of $\ln z$ satisfy the Cauchy–Riemann equations, and find $\frac{\mathrm{d}}{\mathrm{d}z}\ln z$ through the partial derivatives of u and v. (a) use rectangular coordinates, (b) use polar coordinates.

Solution 2.1.3. (a) With rectangular coordinates, $z = x + \mathrm{i}y$,

$$\ln z = u(x,y) + \mathrm{i}v(x,y) = \ln\left(x^2+y^2\right)^{1/2} + \mathrm{i}(\tan^{-1}\frac{y}{x} + 2n\pi).$$

So

$$u = \ln\left(x^2+y^2\right)^{1/2}, \qquad v = (\tan^{-1}\frac{y}{x} + 2n\pi),$$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2}\frac{2x}{(x^2+y^2)} = \frac{x}{(x^2+y^2)}, \\
\frac{\partial u}{\partial y} &= \frac{1}{2}\frac{2y}{(x^2+y^2)} = \frac{y}{(x^2+y^2)}, \\
\frac{\partial v}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{(x^2+y^2)}, \\
\frac{\partial v}{\partial y} &= \frac{1/x}{1+(y/x)^2} = \frac{x}{(x^2+y^2)}.
\end{aligned}$$

Therefore

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

The Cauchy–Riemann equations are satisfied, and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\ln z&=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{x}{(x^2+y^2)}-\mathrm{i}\frac{y}{(x^2+y^2)}\\&=\frac{x-\mathrm{i}y}{(x^2+y^2)}=\frac{x-\mathrm{i}y}{(x+\mathrm{i}y)\,(x-\mathrm{i}y)}=\frac{1}{(x+\mathrm{i}y)}=\frac{1}{z}.\end{aligned}$$

(b) With polar coordinates, $z=r\mathrm{e}^{\mathrm{i}\theta}$,

$$\ln z=u\,(r,\theta)+\mathrm{i}v\,(r,\theta)=\ln r+\mathrm{i}\,(\theta+2n\pi)\,.$$

$$u=\ln\,r,\qquad v=\theta+2n\pi,$$

$$\begin{aligned}\frac{\partial u}{\partial r}&=\frac{1}{r},\qquad &\frac{\partial v}{\partial r}&=0,\\\frac{\partial u}{\partial \theta}&=0,\qquad &\frac{\partial v}{\partial \theta}&=1.\end{aligned}$$

Therefore the Cauchy–Riemann conditions in polar coordinates

$$\frac{\partial u}{\partial r}=\frac{1}{r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\qquad \frac{1}{r}\frac{\partial u}{\partial \theta}=0=-\frac{\partial v}{\partial r},$$

are satisfied. The derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln z=\mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial r}+\mathrm{i}\frac{\partial v}{\partial r}\right)=\mathrm{e}^{-\mathrm{i}\theta}\frac{1}{r}=\frac{1}{r\,\mathrm{e}^{\mathrm{i}\theta}}=\frac{1}{z},$$

as expected.

Example 2.1.4. Show that the real part u and the imaginary part v of z^n satisfy the Cauchy–Riemann equations, and find $\frac{\mathrm{d}}{\mathrm{d}z}z^n$ through the partial derivatives of u and v.

Solution 2.1.4. For this problem, it is much easier to work with polar coordinates with $z=r\mathrm{e}^{\mathrm{i}\theta}$,

$$z^n=u\,(r,\theta)+\mathrm{i}v\,(r,\theta)=r^n\mathrm{e}^{\mathrm{i}n\theta}=r^n\,(\cos n\theta+\mathrm{i}\sin n\theta)\,.$$

$$u=r^n\cos n\theta,\qquad v=r^n\sin n\theta,$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \qquad \frac{\partial u}{\partial \theta} = -nr^{n}\sin n\theta,$$
$$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta, \qquad \frac{\partial v}{\partial \theta} = nr^{n}\cos n\theta.$$

Therefore the Cauchy–Riemann conditions in polar coordinates

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta = \frac{1}{r}\frac{\partial v}{\partial \theta},$$
$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -nr^{n-1}\sin n\theta = -\frac{\partial v}{\partial r},$$

are satisfied and the derivative of z^n is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}z^n &= \mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right) = \mathrm{e}^{-\mathrm{i}\theta}\left(nr^{n-1}\cos n\theta + \mathrm{i}\,nr^{n-1}\sin n\theta\right)\\ &= \mathrm{e}^{-\mathrm{i}\theta}nr^{n-1}\mathrm{e}^{\mathrm{i}n\theta} = nr^{n-1}\mathrm{e}^{\mathrm{i}(n-1)\theta} = n\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)^{n-1} = nz^{n-1},\end{aligned}$$

as one would get regarding z as a single variable.

2.1.6 Analytic Function and Laplace's Equation

Analytic functions have many interesting important properties and applications. One of them is that both the real part and imaginary part of an analytic function satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

A great many physical problems lead to Laplace's equation, naturally we are very much interested in its solution.

If $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is analytic, then u and v satisfy the Cauchy–Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Differentiate the first equation with respect to x and the second equation with respect to y, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Adding the two equations, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}.$$

As long as they are continuous, the order of differentiation can be interchanged

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x},$$

therefore it follows that:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is the Laplace equation for u. Similarly, if we differentiate the first Cauchy–Riemann equation with respect to y, and the second one with respect to x, we can show that v also satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Functions satisfying the Laplace equation are called harmonic functions. Two functions that satisfy both the Laplace equation and the Cauchy–Riemann equations are known as conjugate harmonic functions. We have shown that real and imaginary parts of an analytic function are conjugate harmonic functions.

A family of two-dimensional curves can be represented by the equation

$$u(x, y) = k.$$

For example if $u(x, y) = x^2 + y^2$ and $k = 4$, then this equation represents a circle centered at the origin with radius 2. By changing the constant k, we change the radius of the circle. Thus the equation $x^2 + y^2 = k$ represents a family of circles all centered at the origin with various radii.

Each of the conjugate harmonic functions forming the real and imaginary parts of an analytic function $f(z)$ generates a family of curves in the x–y plane. That is, if $f(z) = u(x, y) + \mathrm{i}v(x, y)$, then $u(x, y) = k$ and $v(x, y) = c$, where k and c are constants, are two families of curves.

If Δu is the difference of u at two nearby points, then by (2.4)

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y.$$

Now if the two points are on the same curve, that is

$$u(x + \Delta x, y + \Delta y) = k, \qquad u(x, y) = k,$$

then

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y) = 0.$$

In this case

$$0 = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y.$$

To find the slope of this curve, we divide both sides of this equation by Δx

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta x},$$

therefore the slope of the curve $u\,(x, y) = k$ is given by

$$\frac{\Delta y}{\Delta x}\,|_u = -\frac{\partial u/\partial x}{\partial u/\partial y}.$$

Similarly, the slope of the curve $v\,(x, y) = c$ is given by

$$\frac{\Delta y}{\Delta x}\,|_v = -\frac{\partial v/\partial x}{\partial v/\partial y}.$$

Since u and v satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the slope of the curve $v\,(x, y) = c$ can be written as

$$\frac{\Delta y}{\Delta x}\,|_v = \frac{\partial u/\partial y}{\partial u/\partial x},$$

which, at any common point, is just the negative reciprocal of the slope of the curve $u\,(x, y) = k$. From the analytic geometry, we know that the two families of curves are orthogonal (perpendicular) to each other. For example, the real part of the analytic function z^2 is $u(x, y) = x^2 - y^2$, the family of curves of $u = k$ is the hyperbolas asymptotic to the line $y = \pm x$ as shown in the z-plane of Fig. 2.1. The imaginary part of z^2 is $v(x, y) = 2xy$, the family of curves of $v = c$ is the hyperbolas asymptotic to the x and y axes, also shown in the z-plane of Fig. 2.1. It is seen that they are indeed orthogonal to each other at the points of intersections.

These remarkable properties of analytic functions serve as basis for many important methods used in fluid dynamics, electrostatics and other branches of physics.

Example 2.1.5. Let $f(z) = u(x, y) + \mathrm{i}\,v(x, y)$ be an analytic function. If $u(x, y) = xy$, find $v(x, y)$ and $f(z)$

Solution 2.1.5.

$$\frac{\partial u}{\partial x}=y=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=x=-\frac{\partial v}{\partial x}.$$

Method 1: Find $f(z)$ from its derivatives

$$\frac{\mathrm{d}f}{\mathrm{d}z}=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=y-\mathrm{i}x=-\mathrm{i}(x+\mathrm{i}y)=-\mathrm{i}z,$$

$$f\,(z)=-\,\mathrm{i}\frac{1}{2}z^2+C.$$

$$f(z)=-\frac{\mathrm{i}}{2}(x+\mathrm{i}y)^2+C=xy-\frac{\mathrm{i}}{2}(x^2-y^2)+C,$$

$$v(x,y)=-\frac{1}{2}(x^2-y^2)+C'.$$

Method 2: Find $v(x,y)$ first

$$\frac{\partial v}{\partial y}=y\qquad\Longrightarrow\qquad v(x,y)=\int y\,\mathrm{d}y=\frac{1}{2}y^2+k(x),$$

$$\frac{\partial v}{\partial x}=-x\qquad\Longrightarrow\qquad\frac{\partial v}{\partial x}=\frac{\mathrm{d}k(x)}{\mathrm{d}x}=-x,\quad\Longrightarrow k(x)=-\frac{1}{2}x^2+C;$$

$$v(x,y)=\frac{1}{2}y^2-\frac{1}{2}x^2+C.$$

$$f(z)=xy+\mathrm{i}\frac{1}{2}\left(y^2-x^2+2C\right),$$

$$x=\frac{1}{2}(z+z^*),\quad y=\frac{1}{2\mathrm{i}}(z-z^*)\quad\text{implies}\quad f(z)=-\frac{1}{2}z^2\mathrm{i}+C'.$$

Example 2.1.6. Let $f(z)=u(x,y)+\mathrm{i}v(x,y)$ be an analytic function. If $u(x,y)=\ln(x^2+y^2)$, find $v(x,y)$ and $f(z)$

Solution 2.1.6.

$$\frac{\partial u}{\partial x}=\frac{2x}{(x^2+y^2)}=\frac{\partial v}{\partial y},\qquad\frac{\partial u}{\partial y}=\frac{2y}{(x^2+y^2)}=-\frac{\partial v}{\partial x}.$$

Method 1: Find $f(z)$ first from its derivatives

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}z} &= \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{2x}{(x^2+y^2)} - \mathrm{i}\frac{2y}{(x^2+y^2)} \\ &= 2\frac{x-\mathrm{i}y}{(x^2+y^2)} = 2\frac{x-\mathrm{i}y}{(x-\mathrm{i}y)(x+\mathrm{i}y)} = 2\frac{1}{x+\mathrm{i}y} = \frac{2}{z},\end{aligned}$$

$$f(z) = 2\ln z + C = \ln z^2 + C,$$

$$z = r\mathrm{e}^{\mathrm{i}\theta}; \qquad r = (x^2+y^2)^{1/2}; \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$\ln z^2 = \ln(x^2+y^2)\mathrm{e}^{\mathrm{i}2\theta} = \ln(x^2+y^2) + \mathrm{i}2\tan^{-1}\frac{y}{x},$$

$$v(x,y) = 2\tan^{-1}\frac{y}{x} + C.$$

Method 2: Find $v(x,y)$ first

$$\frac{\partial v}{\partial y} = \frac{2x}{(x^2+y^2)} \qquad \Longrightarrow \qquad v(x,y) = \int \frac{2x}{(x^2+y^2)}\mathrm{d}y = 2\tan^{-1}\frac{y}{x} + k(x),$$

$$\frac{\partial v(x,y)}{\partial x} = 2\left(\frac{-y}{x^2}\right)\frac{1}{(1+y^2/x^2)} + \frac{\mathrm{d}k(x)}{\mathrm{d}x} = \frac{-2y}{(x^2+y^2)} + \frac{\mathrm{d}k(x)}{\mathrm{d}x},$$

$$\frac{\partial v}{\partial x} = \frac{-2y}{(x^2+y^2)} \qquad \Longrightarrow \frac{\mathrm{d}k(x)}{\mathrm{d}x} = 0, \quad k(x) = C,$$

$$v(x,y) = 2\tan^{-1}\frac{y}{x} + C.$$

$$\begin{aligned}f(z) &= \ln(x^2+y^2) + \mathrm{i}2\tan^{-1}\frac{y}{x} + \mathrm{i}C \\ &= \ln(x^2+y^2)\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{i}C \\ f(z) &= \ln z^2 + C'.\end{aligned}$$

Example 2.1.7. Let $f(z) = \frac{1}{z} = u(x,y) + \mathrm{i}v(x,y)$, (a) show explicitly that the Cauchy–Riemann equations are satisfied; (b) show explicitly that both real part and imaginary part satisfy the Laplace equation; (c) Describe the family of curves $u(x,y) = k$ and $v(x,y) = c$ and sketch them; (d) show explicitly that the curves $u(x,y) = k$ and $v(x,y) = c$ are perpendicular to each other at the points they intersect.

Solution 2.1.7.

$$f(z)=\frac{1}{z}=\frac{1}{x+\mathrm{i}y}=\frac{1}{x+\mathrm{i}y}\cdot\frac{x-\mathrm{i}y}{x-\mathrm{i}y}=\frac{x-\mathrm{i}y}{x^2+y^2}=\frac{x}{x^2+y^2}-\mathrm{i}\frac{y}{x^2+y^2}.$$

Therefore

$$u(x,y)=\frac{x}{x^2+y^2},\qquad v(x,y)=-\frac{y}{x^2+y^2}.$$

(a)

$$\begin{aligned}
\frac{\partial u}{\partial x}&=\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},\\
\frac{\partial u}{\partial y}&=\frac{-2xy}{(x^2+y^2)^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial x}&=\frac{2xy}{(x^2+y^2)^2},\\
\frac{\partial v}{\partial y}&=\frac{-1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}.
\end{aligned}$$

Clearly the Cauchy–Riemann equations are satisfied

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

(b)

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}&=\frac{-2x}{(x^2+y^2)^2}+\frac{(y^2-x^2)(-2)(2x)}{(x^2+y^2)^3}=\frac{2x^3-6xy^2}{(x^2+y^2)^3},\\
\frac{\partial^2 u}{\partial y^2}&=\frac{-2x}{(x^2+y^2)^2}+\frac{-2xy(-2)(2y)}{(x^2+y^2)^3}=\frac{-2x^3+6xy^2}{(x^2+y^2)^3}.
\end{aligned}$$

Thus the real part satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Furthermore,

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2}&=\frac{2y}{(x^2+y^2)^2}+\frac{(2xy)(-2)(2x)}{(x^2+y^2)^3}=\frac{2y^3-6x^2y}{(x^2+y^2)^3},\\
\frac{\partial^2 v}{\partial y^2}&=\frac{2y}{(x^2+y^2)^2}+\frac{(-x^2+y^2)(-2)(2y)}{(x^2+y^2)^3}=\frac{-2y^3+6x^2y}{(x^2+y^2)^3}.
\end{aligned}$$

The imaginary part also satisfies the Laplace equation.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

(c) The equation

$$u(x, y) = \frac{x}{x^2 + y^2} = k$$

can be written as

$$x^2 + y^2 = \frac{x}{k}$$

or

$$\left(x - \frac{1}{2k}\right)^2 + y^2 = \frac{1}{4k^2},$$

which is a circle for any given constant k. Therefore $u(x, y) = k$ is a family of circles centered at $\left(\frac{1}{2k}, 0\right)$ with radius $\frac{1}{2k}$. This family of circles is shown in Fig. 2.5 as solid circles. Similarly

$$v(x, y) = -\frac{y}{x^2 + y^2} = c$$

can be written as

$$x^2 + y^2 = -\frac{y}{c} \qquad \text{or} \qquad x^2 + \left(y + \frac{1}{2c}\right)^2 = \frac{1}{4c^2}.$$

Therefore $v(x, y) = c$ is a family of circles of radius $\frac{1}{2c}$, centered at $(0, -\frac{1}{2c})$. They are shown as the dotted circles in Fig. 2.5.

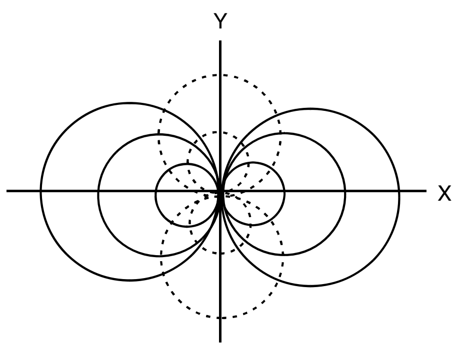


Fig. 2.5. The families of curves described by the real part and imaginary part of the function $f(z) = \frac{1}{z}$

(d) On the curve represented by

$$u(x,y) = \frac{x}{x^2+y^2} = k,$$

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y = 0,$$

which is given by

$$\mathrm{d}u = \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right]\mathrm{d}x - \frac{2xy}{(x^2+y^2)^2}\mathrm{d}y = 0.$$

It follows that:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_u = \frac{y^2-x^2}{2xy}.$$

Similarly, with

$$v(x,y) = -\frac{y}{x^2+y^2} = c,$$

$$\mathrm{d}v = \frac{2xy}{(x^2+y^2)^2}\mathrm{d}x - \left[\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]\mathrm{d}y = 0$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_v = \frac{2xy}{x^2-y^2}.$$

Since the two slopes are negative reciprocals of each other, the two curves are perpendicular.

Those who are familiar with electrostatics will recognize that the curves in Fig. 2.5 are electric field lines and equipotential lines of a line dipole.

2.2 Complex Integration

There are some elegant and powerful theorems regarding integrating analytic functions around a loop. It is these theorems that make complex integrations interesting and useful. But before we discuss these theorems, we must define complex integration.

2.2.1 Line Integral of a Complex Function

When a complex variable z moves in the two-dimensional complex plane, it traces out a curve. Therefore to define a integral of complex function $f(z)$

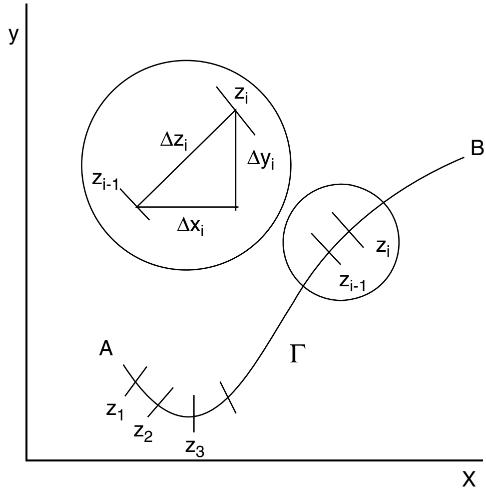


Fig. 2.6. The Riemann sum along the contour Γ which is subdivided into n segments

between two points A and B, we must also specify the path (called contour) along which z moves. The value of the integral will be dependent, in general, upon the contour. However, we will find, that under certain conditions, the integral does not depend upon which of the contours is chosen.

We denote the integral of a complex function $f(z) = u(x,y) + \mathrm{i}v(x,y)$ along a contour Γ from point A to point B as

$$I = \int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z.$$

The integral can be defined in terms of a Riemann sum as in the real variable integration. The contour is subdivided into n segments as shown in Fig. 2.6.

We form the summation

$$I_n = \sum_{i=1}^{n} f(\zeta_i)(z_i - z_{i-1}) = \sum_{i=1}^{n} f(\zeta_i)\,\Delta z_i,$$

where $z_0 = A,\ z_n = B$, and $f(\zeta_i)$ is the function evaluated at a point on Γ between z_{i-1} and z_i. If I_n approaches a limit as $n \to \infty$ and $|\Delta z_i| \to 0$, then we can define the integral as

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \lim_{|\Delta z_i|\to 0,\ n\to\infty} \sum_{i=1}^{n} f(z_i)\,\Delta z_i.$$

Since $\Delta z_i = \Delta x_i + \mathrm{i}\Delta y_i$ as shown in Fig. 2.6, the integral can be written as

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma}^{B} (u+\mathrm{i}v)(\mathrm{d}x+\mathrm{i}\mathrm{d}y) = \int_{A,\Gamma}^{B} [(u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}(v\,\mathrm{d}x + u\,\mathrm{d}y)]$$

$$= \int_{A,\Gamma}^{B} (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\int_{A,\Gamma}^{B} (v\,\mathrm{d}x + u\,\mathrm{d}y). \tag{2.9}$$

Thus the complex contour integral is expressed in terms of two line integrals.

Example 2.2.1. Evaluate the integral $I = \int_A^B z^2\mathrm{d}z$ from $z_A = 0$ to $z_B = 1+\mathrm{i}$, (a) along the contour Γ_1: $y = x^2$, (b) along y-axis from 0 to i, then along the horizontal line from i to $1+\mathrm{i}$, as Γ_2 shown in Fig. 2.7.

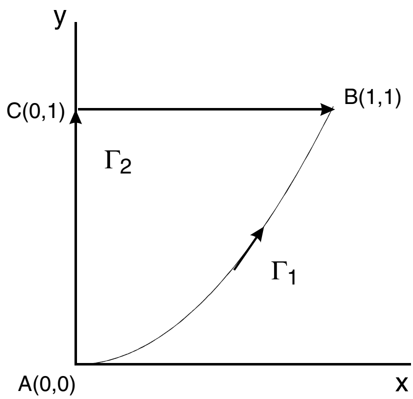


Fig. 2.7. Two contours Γ_1 and Γ_2 from A ($z_A = 0$) to B ($z_B = 1+\mathrm{i}$), Γ_1 : along the curve $y = x^2$, Γ_2 : first along y-axis to C ($z_C = \mathrm{i}$), then along a horizontal line to B

Solution 2.2.1.

$$f(z) = z^2 = (x+\mathrm{i}y)^2 = \left(x^2 - y^2\right) + \mathrm{i}2xy = u + \mathrm{i}v,$$

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma}^{B} [(x^2-y^2)\mathrm{d}x - 2xy\,\mathrm{d}y] + \mathrm{i}\int_{A,\Gamma}^{B} [2xy\,\mathrm{d}x + (x^2-y^2)\mathrm{d}y].$$

(a) Along Γ_1, $\quad y = x^2$, $\mathrm{d}y = 2x\,\mathrm{d}x$,

$$\int_{A,\Gamma_1}^{B} f(z)\,\mathrm{d}z = \int_0^1 [(x^2-x^4)\mathrm{d}x - 2xx^2 2x\,\mathrm{d}x] + \mathrm{i}\int_0^1 [2xx^2\mathrm{d}x + (x^2-x^4)2x\,\mathrm{d}x]$$

$$= \int_0^1 (x^2 - 5x^4)\mathrm{d}x + \mathrm{i}\int_0^1 (4x^3 - 2x^5)\mathrm{d}x = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.$$

(b) Let $z_C = \mathrm{i}$ as shown in Fig. 2.7. So

$$\int_{A,\Gamma_2}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma_2}^{C} f(z)\,\mathrm{d}z + \int_{C,\Gamma_2}^{B} f(z)\,\mathrm{d}z.$$

From A to C : $x = 0, \mathrm{d}x = 0$

$$\int_{A,\Gamma_2}^{C} f(z)\,\mathrm{d}z = \mathrm{i}\int_0^1 (-y^2)\mathrm{d}y = -\frac{1}{3}\mathrm{i}.$$

From C to B : $y = 1,\ \mathrm{d}y = 0$

$$\int_{C,\Gamma_2}^{B} f(z)\,\mathrm{d}z = \int_0^1 (x^2 - 1)\mathrm{d}x + \mathrm{i}\int_0^1 2x\,\mathrm{d}x = -\frac{2}{3} + \mathrm{i}.$$

$$\int_{A,\Gamma_2}^{B} f(z)\,\mathrm{d}z = -\frac{1}{3}\mathrm{i} - \frac{2}{3} + \mathrm{i} = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.$$

The integrals along Γ_1 and Γ_2 are observed to be equal.

2.2.2 Parametric Form of Complex Line Integral

If along the contour Γ, z is expressed parametrically, these line integrals can be transformed into ordinary integrals in which there is only one independent variable. For if $z = z(t)$, where t is a parameter, and $A = z(t_A)$, $B = z(t_B)$, then

$$\int_A^B f(z)\,\mathrm{d}z = \int_{t_A}^{t_B} f(z(t))\,\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t. \tag{2.10}$$

For instance, on Γ_1 of the previous example, $y = x^2$, we can set $z(t) = x(t) + \mathrm{i}y(t)$ with $x(t) = t$ and $y(t) = t^2$. It follows that $\frac{\mathrm{d}z}{\mathrm{d}t} = 1 + \mathrm{i}2t$, and

$$\begin{aligned}\int_{A,\Gamma_1}^{B} z^2\mathrm{d}z &= \int_0^1 \left(t + \mathrm{i}t^2\right)^2 (1 + \mathrm{i}2t)\,\mathrm{d}t \\ &= \int_0^1 [(t^2 - 5t^4) + \mathrm{i}(4t^3 - 2t^5)]\mathrm{d}t = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.\end{aligned}$$

Similarly, on Γ_2 of the previous example, from A to C, we can set $z(t) = \mathrm{i}t$ with $0 \le t \le 1$, and $\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}$. From C to B, we can set $z(t) = (t-1) + \mathrm{i}$ with $1 \le t \le 2$, and $\frac{\mathrm{d}z}{\mathrm{d}t} = 1$. Thus

$$\begin{aligned}\int_{A,\Gamma_2}^{B} z^2\mathrm{d}z &= \int_0^1 (\mathrm{i}t)^2\,\mathrm{i}\mathrm{d}t + \int_1^2 (t - 1 + \mathrm{i})^2\,\mathrm{d}t \\ &= -\frac{1}{3}\mathrm{i} - \frac{2}{3} + \mathrm{i} = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.\end{aligned}$$

Parametrization of a Circular Contour

A circular contour can be easily parameterized with the angular variable of the polar coordinates. This is of considerable importance because through the principle of deformation of contours, which we will soon see, other contour integrations can also be carried out by changing the contour into a circle.

Consider the integral $I = \oint_C f(z)\,\mathrm{d}z$, where C is a circle of radius r centered at the origin. Clearly we can express z as

$$z(\theta) = r\cos\theta + \mathrm{i}\,r\sin\theta = r\,\mathrm{e}^{\mathrm{i}\theta},$$

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -r\sin\theta + \mathrm{i}\,r\cos\theta = \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}.$$

This means $\mathrm{d}z = \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$, so the integral becomes

$$I = \int_0^{2\pi} f\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta.$$

The following example will illustrate how this is done.

Example 2.2.2. Evaluate the integral $\oint_C z^n\mathrm{d}z$, where n is an integer and C is a circle of radius r around the origin.

Solution 2.2.2.

$$\oint_C z^n\mathrm{d}z = \int_0^{2\pi}\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)^n\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}\,r^{n+1}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n+1)\theta}\mathrm{d}\theta.$$

For $n \neq -1$

$$\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n+1)\theta}\mathrm{d}\theta = \frac{1}{\mathrm{i}(n+1)}\left[\mathrm{e}^{\mathrm{i}(n+1)\theta}\right]_0^{2\pi} = \frac{1}{\mathrm{i}(n+1)}[1-1] = 0.$$

For $n = -1$

$$\int_0^{2\pi}\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)^n\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi}\mathrm{d}\theta = 2\pi\mathrm{i}.$$

This means

$$\oint_C z^n\mathrm{d}z = 0 \quad \text{for} \quad n \neq -1,$$

$$\oint_C \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}.$$

Note that these results are independent of the radius r.

Some Properties of Complex Line Integral

The parametric form of the complex line integral enables us to see immediately that many formulas of ordinary integration of real variables can be directly applied to the complex integration. For example, the complex integral from B to A along the same path Γ is given by the right-hand side of (2.10) with t_A and t_B interchanged, introducing a negative sign to the equation. Therefore

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = -\int_{B,\Gamma}^{A} f(z)\,\mathrm{d}z.$$

Similarly, if C is on Γ, then

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma}^{C} f(z)\,\mathrm{d}z + \int_{C,\Gamma}^{B} f(z)\,\mathrm{d}z.$$

If the integral from A to B is along Γ_1 and from B back to A is along a different contour Γ_2, we can write the sum of the two integrals as

$$\int_{A,\Gamma_1}^{B} f(z)\,\mathrm{d}z + \int_{B,\Gamma_2}^{A} f(z)\,\mathrm{d}z = \oint_{\Gamma} f(z)\,\mathrm{d}z,$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and the symbol $\oint_{\Gamma}$ is to signify that the integration is taken counterclockwise along the closed contour Γ. Thus

$$\begin{aligned}\oint_{\text{c.c.w.}} f(z)\,\mathrm{d}z &= \int_{A,\Gamma_1}^{B} f(z)\,\mathrm{d}z + \int_{B,\Gamma_2}^{A} f(z)\,\mathrm{d}z \\ &= -\int_{B,\Gamma_1}^{A} f(z)\,\mathrm{d}z - \int_{A,\Gamma_2}^{B} f(z)\,\mathrm{d}z = -\oint_{\text{c.w.}} f(z)\,\mathrm{d}z,\end{aligned}$$

where c.c.w. means counterclockwise and c.w. means clockwise.

Furthermore, we can show that

$$\left|\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z\right| \le ML, \tag{2.11}$$

where M is the maximum value of $|f(z)|$ on Γ and L is the length of Γ. This is because

$$\left|\int_{t_A}^{t_B} f(z)\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t\right| \le \int_{t_A}^{t_B}\left|f(z)\frac{\mathrm{d}z}{\mathrm{d}t}\right|\mathrm{d}t,$$

which is a generalization of $|z_1 + z_2| \le |z_1| + |z_2|$. By the definition of M, we have

$$\int_{t_A}^{t_B}\left|f(z)\frac{\mathrm{d}z}{\mathrm{d}t}\right|\mathrm{d}t \le M\int_{t_A}^{t_B}\left|\frac{\mathrm{d}z}{\mathrm{d}t}\right|\mathrm{d}t = M\int_{A}^{B}|\mathrm{d}| = ML.$$

Thus, starting with (2.10), we have

$$\left|\int_{A}^{B} f(z)\,\mathrm{d}z\right| = \left|\int_{t_A}^{t_B} f(z)\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t\right| \le ML.$$

2.3 Cauchy's Integral Theorem

As we have seen, the results of integrations of z^2 along Γ_1 and Γ_2 of Fig. 2.7 are exactly the same. Therefore a closed loop integration from A to B along Γ_1 and returning from B to A along Γ_2 is equal to zero. In 1825, Cauchy proved a theorem which enables use to see that this must be the case without carrying out the integration. Before we discuss this theorem, let us first review the Green's lemma of real variables.

2.3.1 Green's Lemma

There is an important relation that allows us to transform a line integral into an area integral for lines and areas in the xy plane. It is often referred to as Green's lemma, which states that

$$\oint_C [P(x,y)dx + Q(x,y)\mathrm{d}y] = \iint_R \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y, \qquad (2.12)$$

where C is a closed curve surrounding the region R. The curve C is traversed counterclockwise, that is with the region R always to the left as shown in Fig. 2.8.

To prove Green's lemma, let us use Fig. 2.9, part (a) to carry out the first part of the area double integral

$$\begin{aligned}\iint_R \frac{\partial Q(x,y)}{\partial x}\mathrm{d}x\,\mathrm{d}y &= \int_c^d \left[\int_{x=g_1(y)}^{x=g_2(y)} \frac{\partial Q(x,y)}{\partial x}\mathrm{d}x\right]\mathrm{d}y \\ &= \int_c^d [Q(x,y)]_{x=g_1(y)}^{x=g_2(y)}\,\mathrm{d}y.\end{aligned}$$

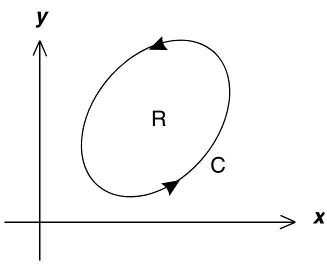


Fig. 2.8. The closed contour C of the line integral in the Green's lemma. C is counterclockwise and is defined as the positive direction with respect to the interior of R

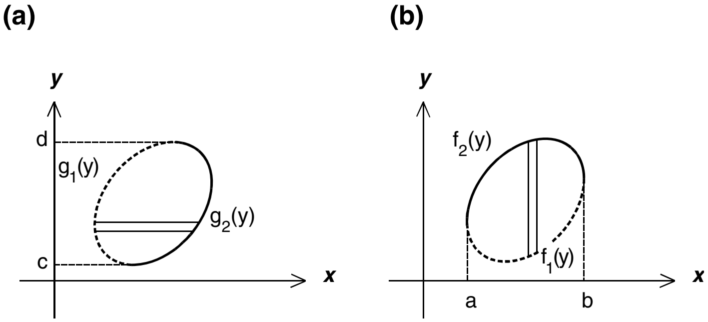


Fig. 2.9. Same contour but with two different ways to carry out the area double integral in the Green's lemma

Now

$$\int_c^d [Q(x,y)]_{x=g_1(y)}^{x=g_2(y)}\,\mathrm{d}y = \int_c^d Q(g_2(y),y)\mathrm{d}y - \int_c^d Q(g_1(y),y)\mathrm{d}y$$
$$= \int_c^d Q(g_2(y),y)\mathrm{d}y + \int_d^c Q(g_1(y),y)\mathrm{d}y.$$

The contour of the last line integral is from $y=c$ going through $g_2(y)$ to $y=d$ and then returning through $g_1(y)$ to $y=c$. Clearly it is counterclockwise closed loop integral

$$\iint_R \frac{\partial Q(x,y)}{\partial x}\mathrm{d}x\,\mathrm{d}y = \oint_{\text{c.c.w}} Q\,(x,y)\,\mathrm{d}y. \tag{2.13}$$

Next we will use Fig. 2.9, part (b) to carry out the second part of the area double integral

$$\iint_R \frac{\partial P(x,y)}{\partial y}\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial P(x,y)}{\partial y}\mathrm{d}y\right]\mathrm{d}x = \int_a^b [P(x,y)]_{y=f_1(x)}^{y=f_2(x)}\,\mathrm{d}x.$$

$$\int_a^b [P(x,y)]_{y=f_1(x)}^{y=f_2(x)}\,\mathrm{d}x = \int_a^b P(x,f_2(x))\mathrm{d}x - \int_a^b P(x,f_1(x))\mathrm{d}x$$
$$= \int_a^b P(x,f_2(x))\mathrm{d}x + \int_b^a P(x,f_1(x))\mathrm{d}x.$$

In this case the contour is from $x=a$ going through $f_2(x)$ to $x=b$ and then returning to $x=a$ through $f_1(x)$. Therefore it is clockwise

$$\iint_R \frac{\partial P(x,y)}{\partial y}\mathrm{d}x\,\mathrm{d}y = \oint_{\text{c.w.}} P(x,y)\mathrm{d}x = -\oint_{\text{c.c.w.}} P(x,y)\mathrm{d}x. \tag{2.14}$$

In the last step we changed the sign to make it counterclockwise.

Subtracting (2.14) from (2.13), we have the Green's lemma

$$\iint_R \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y = \oint_{\text{c.c.w.}} [Q(x,y)\mathrm{d}y + P(x,y)\mathrm{d}x].$$

2.3.2 Cauchy–Goursat Theorem

An important theorem in complex integration is the following:

If C is a closed contour and $f(z)$ is analytic on and inside C, then

$$\oint_C f(z)\mathrm{d}z = 0. \tag{2.15}$$

This is known as Cauchy's theorem. The proof goes as follows. Starting with

$$\oint_C f(z)\mathrm{d}z = \oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\oint_C (v\,\mathrm{d}x + u\,\mathrm{d}y), \tag{2.16}$$

making use of the Green's lemma of (2.12) and identifying P as u and Q as $-v$, we have

$$\oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) = \int\int_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y.$$

Since $f(z)$ is analytic, so u and v satisfy Cauchy–Riemann conditions. In particular

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y},$$

therefore the area double integral is equal to zero, thus

$$\oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) = 0.$$

Similarly, identifying u as Q and v as P, from Green's lemma we have

$$\oint_C (v\,\mathrm{d}x + u\,\mathrm{d}y) = \int\int_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y.$$

Because of the other Cauchy–Riemann condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

the integral on the left-hand side is also equal to zero

$$\oint_C (v\,\mathrm{d}x + u\,\mathrm{d}y) = 0.$$

Thus both line integrals on the right-hand side of (2.16) are zero, therefore

$$\oint_C f(z)\,\mathrm{d}z = 0,$$

which is known as Cauchy's integral theorem.

In this proof, we have used Green's lemma which requires the first partial derivatives of u and v to be continuous. Therefore we have implicitly assumed that the derivative of $f(z)$ is continuous. In 1903, Goursat proved this theorem without assuming the continuity of $f'(z)$. Therefore this theorem is also called Cauchy–Goursat theorem. Mathematically Goursat's removal of the continuity assumption from the proof of the theorem is very important because it enables us to rigorously establish that derivatives of analytic functions are analytic, and they are automatically continuous. A version of Goursat's proof can be found in *Complex Variables and Applications,* by J.W. Brown and R.V. Churchill *Complex Variable and Applications,* 5th edn. (McGraw-Hill, New York 1989).

2.3.3 Fundamental Theorem of Calculus

If the closed contour Γ is divided into two parts Γ_1 and Γ_2, as shown in Fig. 2.7, and $f(z)$ is analytic on and between Γ_1 and Γ_2, then Cauchy's integral theorem can be written as

$$\begin{aligned}\oint_\Gamma f(z)\mathrm{d}z &= \int_{A\ \Gamma_1}^{B} f(z)\mathrm{d}z + \int_{B\ \Gamma_2}^{A} f(z)\mathrm{d}z \\ &= \int_{A\ \Gamma_1}^{B} f(z)\mathrm{d}z - \int_{A\ \Gamma_2}^{B} f(z)\mathrm{d}z = 0,\end{aligned}$$

where the negative sign appears since we have exchanged the limit on the last integral. Thus we have

$$\int_{A\ \Gamma_1}^{B} f(z)\mathrm{d}z = \int_{A\ \Gamma_2}^{B} f(z)\mathrm{d}z, \tag{2.17}$$

showing that the value of a line integral between two points is independent of the path provided that the integrand is an analytic function in the domain on and between the contours.

With this in mind, we can show that, as long as $f(z)$ is analytic in a region containing A and B

$$\int_A^B f(z)\,\mathrm{d}z = F(B) - F(A),$$

where

$$\frac{\mathrm{d}F(z)}{\mathrm{d}z} = \lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)\,.$$

The integral

$$F(z) = \int_{z_0}^{z} f(z')\,\mathrm{d}z' \tag{2.18}$$

uniquely define the function $F(z)$ if z_0 is a fixed point and $f(z')$ is analytic throughout the region containing the path between z_0 and z. Similarly, we can define

$$F(z+\Delta z) = \int_{z_0}^{z+\Delta z} f(z')\,\mathrm{d}z' = \int_{z_0}^{z} f(z')\,\mathrm{d}z' + \int_{z}^{z+\Delta z} f(z')\,\mathrm{d}z'.$$

Clearly

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(z')\,\mathrm{d}z'.$$

For a small Δz, the right-hand side reduces to

$$\int_{z}^{z+\Delta z} f(z')\,\mathrm{d}z' \to f(z)\,\Delta z,$$

which implies that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z).$$

Thus

$$\frac{\mathrm{d}F(z)}{\mathrm{d}z} = f(z)$$

and the fundamental theorem of calculus follows:

$$\int_{A}^{B} f(z)\,\mathrm{d}z = \int_{A}^{B} \mathrm{d}F(z) = F(B) - F(A).$$

Example 2.3.1. Find the value of the integral $\int_0^{1+\mathrm{i}} z^2\,\mathrm{d}z$.

Solution 2.3.1.

$$\int_0^{1+\mathrm{i}} z^2\mathrm{d}z = \left[\frac{1}{3}z^3\right]_0^{1+\mathrm{i}} = \frac{1}{3}(1+\mathrm{i})^3 = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.$$

Note that the result is the same as in Example 2.2.1.

Example 2.3.2. Find the values of the following integrals:

$$I_1 = \int_{-\pi\mathrm{i}}^{\pi\mathrm{i}} \cos z\,\mathrm{d}z, \qquad I_2 = \int_{4+\pi\mathrm{i}}^{4-3\pi\mathrm{i}} \mathrm{e}^{z/2}\mathrm{d}z.$$

Solution 2.3.2.

$$I_1 = \int_{-\pi\mathrm{i}}^{\pi\mathrm{i}} \cos z\, \mathrm{d}z = [\sin z]_{-\pi\mathrm{i}}^{\pi\mathrm{i}} = \sin(\pi\mathrm{i}) - \sin(-\pi\mathrm{i})$$
$$= 2\sin(\pi\mathrm{i}) = 2\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}(\mathrm{i}\pi)} - \mathrm{e}^{-\mathrm{i}(\mathrm{i}\pi)}\right) = \left(\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}\right)\mathrm{i} \simeq 23.097\mathrm{i}.$$

$$I_2 = \int_{4+\pi\mathrm{i}}^{4-3\pi\mathrm{i}} \mathrm{e}^{z/2}\mathrm{d}z = \left[2\mathrm{e}^{z/2}\right]_{4+\pi\mathrm{i}}^{4-3\pi\mathrm{i}} = 2\left(\mathrm{e}^{2-\mathrm{i}3\pi/2} - \mathrm{e}^{2+\mathrm{i}\pi/2}\right)$$
$$= 2\mathrm{e}^2\left(\mathrm{e}^{-\mathrm{i}3\pi/2} - \mathrm{e}^{\mathrm{i}\pi/2}\right) = 2\mathrm{e}^2(\mathrm{i}-\mathrm{i}) = 0.$$

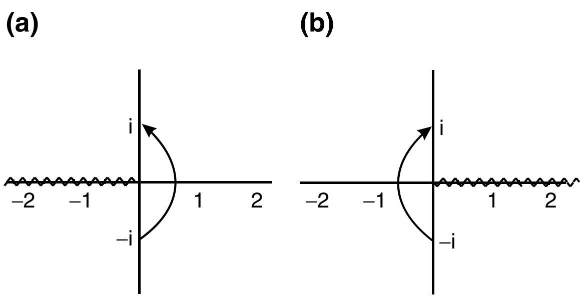
Example 2.3.3. Find the values of the following integral:

$$\int_{-\mathrm{i}}^{\mathrm{i}} \frac{\mathrm{d}z}{z}.$$

Solution 2.3.3. Since $z = 0$ is a singular point, the path of integration must not pass through the origin. Furthermore

$$\int \frac{\mathrm{d}z}{z} = \ln z + C,$$

where $\ln z$ is a multivalued function, therefore there is a branch cut. To evaluate this definite integral we must specify the path of z going from $-\mathrm{i}$ to i. There are two possibilities as shown in (a) and (b) of the following figure:



(a) To go from $-\mathrm{i}$ to i in the right half of the complex plane, we must take the negative real axis as the branch cut. In the principal branch, $-\pi < \theta < \pi$. Thus

$$\int_{-\mathrm{i}}^{\mathrm{i}} \frac{\mathrm{d}z}{z} = [\ln z]_{-\mathrm{i}}^{\mathrm{i}} = \left[\ln \mathrm{e}^{\mathrm{i}\theta}\right]_{\theta=-\frac{1}{2}\pi}^{\theta=\frac{1}{2}\pi} = [\mathrm{i}\theta]_{\theta=-\frac{1}{2}\pi}^{\theta=\frac{1}{2}\pi} = \mathrm{i}\frac{1}{2}\pi + \mathrm{i}\frac{1}{2}\pi = \mathrm{i}\pi.$$

(b) To go from $-\mathrm{i}$ to i in the left half of the complex plane, we must take the positive real axis as the branch cut. Therefore $0 < \theta < 2\pi$. Thus

$$\int_{-\mathrm{i}}^{\mathrm{i}} \frac{\mathrm{d}z}{z} = [\ln z]_{-\mathrm{i}}^{\mathrm{i}} = \left[\ln \mathrm{e}^{\mathrm{i}\theta}\right]_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} = [\mathrm{i}\theta]_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} = \mathrm{i}\frac{1}{2}\pi - \mathrm{i}\frac{3}{2}\pi = -\mathrm{i}\pi.$$

2.4 Consequences of Cauchy's Theorem

2.4.1 Principle of Deformation of Contours

There is an immediate, practical consequence of the Cauchy Integral Theorem. The contour of a complex integral can be arbitrarily deformed through an analytic region without changing the value of the integral.

Consider the integration along the two contours shown on the left side of Fig. 2.10. If $f(z)$ is analytic, then

$$\oint_{abcda} f(z)\mathrm{d}z = 0, \tag{2.19}$$

$$\oint_{efghe} f(z)\mathrm{d}z = 0. \tag{2.20}$$

Naturally the sum of them is also equal to zero

$$\oint_{abcda} f(z)\mathrm{d}z + \oint_{efghe} f(z)\mathrm{d}z = 0. \tag{2.21}$$

Notice that the integrals along ab and along he are in the opposite direction. If ab coincides with he, their contributions will cancel each other. Thus if the gaps between ab and he, and between cd and fg are shrinking to zero, the sum of these two integrals becomes the sum of the integral along the outer

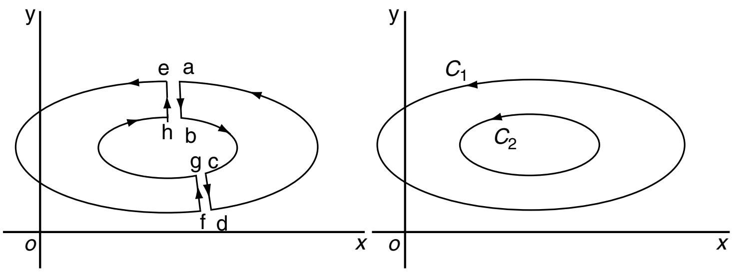


Fig. 2.10. Contour deformation

contour C_1 and the integral along the inner contour C_2 but in the opposite direction. If we change the direction of C_2, we must change the sign of the integral. Therefore

$$\oint\limits_{abcda} f(z)\mathrm{d}z + \oint\limits_{efghe} f(z)\mathrm{d}z = \oint\limits_{C_1} f(z)\mathrm{d}z - \oint\limits_{C_2} f(z)\mathrm{d}z = 0.$$

It follows:

$$\oint\limits_{C_1} f(z)\mathrm{d}z = \oint\limits_{C_2} f(z)\mathrm{d}z. \tag{2.22}$$

Thus we have shown that the line integral of an analytic function around any closed curve C_1 is equal to the line integral of the same function around any other closed curve C_2 into which C_1 can be continuously deformed as long as $f(z)$ is analytic between C_1 and C_2 and is single-valued on C_1 and C_2.

2.4.2 The Cauchy Integral Formula

The Cauchy integral formula is a natural extension of the Cauchy integral theorem. Consider the integral

$$I_1 = \oint_{C_1} \frac{f(z)}{z - z_0}\mathrm{d}z, \tag{2.23}$$

where $f(z)$ is analytic everywhere in the z-plane, and C_1 is a closed contour that does not include the point z_0 as shown in Fig. 2.11a.

Since $(z - z_0)^{-1}$ is analytic everywhere except at $z = z_0$, and z_0 is outside of C_1, therefore $f(z)/(z - z_0)$ is analytic inside C_1. It follows from Cauchy's integral theorem that

$$I_1 = \oint_{C_1} \frac{f(z)}{z - z_0}\mathrm{d}z = 0. \tag{2.24}$$

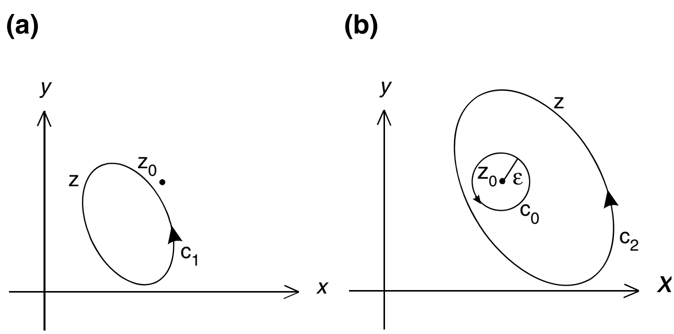


Fig. 2.11. Closed contour integration. (**a**) The singular point z_0 is outside of the contour C_1. (**b**) The contour C_2 encloses z_0, C_2 can be deformed into the circle C_0 without changing the value of the integral

Now consider a second integral

$$I_2 = \oint_{C_2} \frac{f(z)}{z - z_0} \mathrm{d}z, \tag{2.25}$$

similar to the first, except now the contour C_2 encloses z_0, as shown in Fig. 2.11b. The integrand in this integral is not analytic at $z = z_0$ which is inside C_2, so we cannot invoke the Cauchy integral theorem to argue that $I_2 = 0$. However, the integrand is analytic everywhere, except at the point $z = z_0$, so we can deform the contour into an infinitesimal circle of radius ε centered at z_0, without changing its value

$$I_2 = \lim_{\varepsilon \to 0} \oint_{C_0} \frac{f(z)}{z - z_0} \mathrm{d}z. \tag{2.26}$$

This deformation is also shown in Fig. 2.11b.

This last integral can be evaluated. In order to see more clearly, we enlarge the contour in Fig. 2.12.

Since z is on the circle C_0, with the notation shown in Fig. 2.12, it is clear that

$$\begin{aligned} z &= x + \mathrm{i}y, \\ x &= x_0 + \varepsilon \cos\theta, \\ y &= y_0 + \varepsilon \sin\theta. \end{aligned}$$

Therefore

$$z = (x_0 + \mathrm{i}y_0) + \varepsilon(\cos\theta + \mathrm{i}\sin\theta). \tag{2.27}$$

Since

$$\begin{aligned} z_0 &= x_0 + \mathrm{i}y_0, \\ \mathrm{e}^{\mathrm{i}\theta} &= \cos\theta + \mathrm{i}\sin\theta, \end{aligned}$$

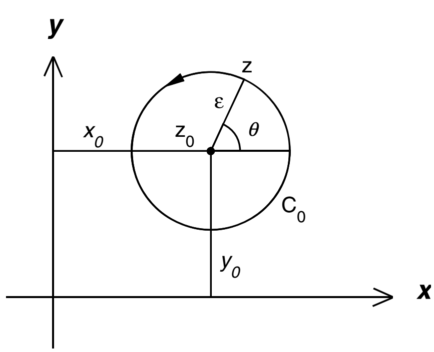


Fig. 2.12. Circular contour for the Cauchy integral formula

we can write

$$z = z_0 + \varepsilon e^{i\theta}. \tag{2.28}$$

On C_0, ε is a constant, and θ goes from 0 to 2π. Therefore

$$dz = i\,\varepsilon\, e^{i\theta} d\theta \tag{2.29}$$

and

$$\oint_{C_0} \frac{f(z)}{z - z_0} dz = \int_0^{2\pi} \frac{f\left(z_0 + \varepsilon\, e^{i\theta}\right)}{\varepsilon\, e^{i\theta}} i\,\varepsilon\, e^{i\theta} d\theta = i \int_0^{2\pi} f\left(z_0 + \varepsilon\, e^{i\theta}\right) d\theta. \tag{2.30}$$

As $\varepsilon \longrightarrow 0$, $f\left(z_0 + \varepsilon\, e^{i\theta}\right) \to f(z_0)$ and can be taken outside the integral

$$\begin{aligned} I_2 &= \oint_{C_2} \frac{f(z)}{z - z_0} dz = \lim_{\varepsilon \to 0} \oint_{C_0} \frac{f(z)}{z - z_0} dz \\ &= \lim_{\varepsilon \to 0} i \int_0^{2\pi} f\left(z_0 + \varepsilon\, e^{i\theta}\right) d\theta = i\, f(z_0) \int_0^{2\pi} d\theta = 2\pi i\, f(z_0), \end{aligned} \tag{2.31}$$

where C is any closed, counterclockwise path that encloses z_0, and $f(z)$ is analytic inside C. This result is known as Cauchy's integral formula, usually written as

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz. \tag{2.32}$$

2.4.3 Derivatives of Analytic Function

If we differentiate both sides of Cauchy's integral formula, interchanging the order of differentiation and integration, we get

$$f'(z_0) = \frac{1}{2\pi i} \oint_C f(z) \frac{d}{dz_0} \frac{1}{(z - z_0)} dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz.$$

To establish this formula in a rigorous manner, we may start with the formal expression of the derivative

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z_0 \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0} \left[f(z_0 + \Delta z_0) - f(z_0)\right] \\ &= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0} \left[\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0 - \Delta z_0} dz - \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz\right]. \end{aligned}$$

Now

$$\begin{aligned} &\oint_C \frac{f(z)}{z - z_0 - \Delta z_0} dz - \oint_C \frac{f(z)}{z - z_0} dz = \oint_C f(z) \left(\frac{1}{z - z_0 - \Delta z_0} - \frac{1}{z - z_0}\right) dz \\ &\quad = \oint_C f(z) \frac{\Delta z_0}{(z - z_0 - \Delta z_0)(z - z_0)} dz = \Delta z_0 \oint_C \frac{f(z)dz}{(z - z_0 - \Delta z_0)(z - z_0)}. \end{aligned}$$

Therefore

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0} \left[\frac{\Delta z_0}{2\pi\mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{(z - z_0 - \Delta z_0)(z - z_0)} \right] \\ &= \lim_{\Delta z_0 \to 0} \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{(z - z_0 - \Delta z_0)(z - z_0)} = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^2}\mathrm{d}z. \end{aligned}$$

In a like manner we can show that

$$f''(z_0) = \frac{2}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^3}\mathrm{d}z \tag{2.33}$$

and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\mathrm{d}z. \tag{2.34}$$

Thus we have established the fact that analytic functions possess derivatives of all orders. Also, all derivatives of analytic functions are analytic. This is quite different from our experience with real variables, where we have encountered functions that possess first and second derivatives at a particular point, but yet the third derivative is not defined.

Cauchy's integral formula allows us to determine the value of an analytic function at any point z interior to a simply connected region by integrating around a curve C surrounding the region. Only values of the function on the boundary are used. Thus, we note that if an analytic function is prescribed on the entire boundary of a simply connected region, the function and all its derivatives can be determined at all interior points. The Cauchy's integral formula can be written in the form of

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(\varsigma)}{\varsigma - z}\mathrm{d}\varsigma, \tag{2.35}$$

where z is any interior point inside C. The complex variable ς is on C and is simply a dummy variable of integration that disappears in the integration process. Cauchy's integral formula is often used in this form.

Example 2.4.1. Evaluate the integrals

$$\text{(a)} \oint \frac{z^2 \sin \pi z}{z - \frac{1}{2}}\mathrm{d}z, \qquad \text{(b)} \oint \frac{\cos z}{z^3}\mathrm{d}z$$

around the circle $|z| = 1$.

Solution 2.4.1. (a) The singular point is at $z = \frac{1}{2}$ which is inside the circle $|z| = 1$. Therefore

$$\oint \frac{z^2 \sin \pi z}{z - \frac{1}{2}}\mathrm{d}z = 2\pi\mathrm{i} \left[z^2 \sin \pi z \right]_{z=1/2} = 2\pi\mathrm{i} \left(\frac{1}{2} \right)^2 \sin\left(\pi \frac{1}{2} \right) = \frac{1}{2}\pi\mathrm{i}.$$

(b) The singular point is at $z = 0$ which is inside the circle $|z| = 1$. Therefore

$$\oint \frac{\cos z}{z^3} \mathrm{d}z = \frac{2\pi \mathrm{i}}{2!} \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} \cos z \right]_{z=0} = \pi \mathrm{i}[-\cos(0)] = -\pi \mathrm{i}.$$

Example 2.4.2. Evaluate the integral

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z$$

around (a) the circle $|z| = 1$, (b) the circle $|z| = 3$.

Solution 2.4.2. (a) The singular point is at $z = 2$. It is outside the circle of $|z| = 1$, as shown in Fig. 2.13a. Inside the circle $|z| = 1$, the function $\frac{z^2-1}{(z-2)^2}$ is analytic, therefore

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z = 0.$$

(b) Since $z = 2$ is inside the circle $|z| = 3$, as shown in Fig. 2.13b, we can write the integral as

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z = \oint \frac{f(z)}{(z-2)^2} \mathrm{d}z = 2\pi \mathrm{i} \, f'(2),$$

where

$$f(z) = z^2 - 1, \qquad f'(z) = 2z, \qquad \text{and} \qquad f'(2) = 4.$$

Thus

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z = 2\pi \mathrm{i} 4 = 8\pi \mathrm{i}.$$

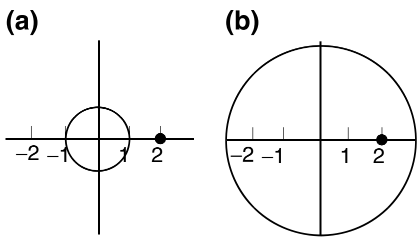


Fig. 2.13. (**a**) $|z| = 1$, (**b**) $|z| = 3$

Example 2.4.3. Evaluate the integral

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z$$

(a) around the circle $|z-1|=1$, (b) around the circle $|z-\mathrm{i}|=1$, (c) around the circle $|z-1|=2$.

Solution 2.4.3. Unless the relationship between the singular points and the contour is clear as in previous examples, to solve problems of closed contour integration, it is best to first find the singular points (known as poles) and display them on the complex plane, then draw the contour. In this particular problem, the singular points are at $z=\pm\mathrm{i}$, which are the solutions of $z^2+1=0$. The three contours are shown in Fig. 2.14.

(a) It is seen that both singular points are outside of the contour $|z-1|=1$, therefore

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z=0.$$

(b) In this case, only one singular point $z=i$ is inside the contour, so we can write

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z=\oint \frac{z^2}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z=\oint \frac{f(z)}{z-\mathrm{i}}\mathrm{d}z,$$

where

$$f(z)=\frac{z^2}{z+\mathrm{i}}.$$

Thus, it follows:

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z=\oint \frac{f(z)}{z-\mathrm{i}}\mathrm{d}z=2\pi\mathrm{i}\,f(\mathrm{i})=2\pi\mathrm{i}\frac{(\mathrm{i})^2}{\mathrm{i}+\mathrm{i}}=-\pi.$$

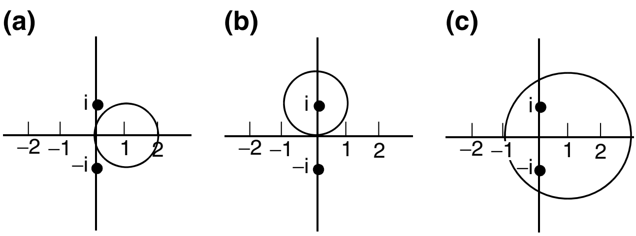


Fig. 2.14. (**a**) $|z-1|=1$, (**b**) $|z-\mathrm{i}|=1$, (**c**) $|z-1|=2$

(c) In this case, both singular points are inside the contour. To make use of the Cauchy integral formula, we first take the partial fraction of $\frac{1}{z^2+1}$,

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{A}{(z-\mathrm{i})} + \frac{B}{(z+\mathrm{i})}\\ &= \frac{A(z+\mathrm{i})+B(z-\mathrm{i})}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{(A+B)z+(A-B)\mathrm{i}}{(z-\mathrm{i})(z+\mathrm{i})}.\end{aligned}$$

So

$$\begin{gathered}A+B=0, \quad (A-B)\mathrm{i}=1,\\ B=-A, \qquad 2A\mathrm{i}=1,\\ A=\frac{1}{2\mathrm{i}}=-\frac{\mathrm{i}}{2}, \quad B=\frac{\mathrm{i}}{2}.\end{gathered}$$

It follows that:

$$\begin{aligned}\oint \frac{z^2}{z^2+1}\mathrm{d}z &= \oint z^2\left(-\frac{\mathrm{i}}{2}\frac{1}{(z-\mathrm{i})} + \frac{\mathrm{i}}{2}\frac{1}{(z+\mathrm{i})}\right)\mathrm{d}z\\ &= -\frac{\mathrm{i}}{2}\oint \frac{z^2}{z-\mathrm{i}}\mathrm{d}z + \frac{\mathrm{i}}{2}\oint \frac{z^2}{z+\mathrm{i}}\mathrm{d}z.\end{aligned}$$

Each integral on the right-hand side has only one singular point inside the contour. According the Cauchy integral formula

$$\begin{aligned}\oint \frac{z^2}{z-\mathrm{i}}\mathrm{d}z &= 2\pi\mathrm{i}(\mathrm{i})^2 = -2\pi\mathrm{i},\\ \oint \frac{z^2}{z+\mathrm{i}}\mathrm{d}z &= 2\pi\mathrm{i}\,(-\mathrm{i})^2 = -2\pi\mathrm{i}.\end{aligned}$$

Therefore

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z = -\frac{\mathrm{i}}{2}\,(-2\pi\mathrm{i}) + \frac{\mathrm{i}}{2}\,(-2\pi\mathrm{i}) = 0.$$

Example 2.4.4. Evaluate the integral

$$\oint \frac{z-1}{2z^2+3z-2}\mathrm{d}z$$

around the square whose vertices are $(1,1),\ (-1,1),\ (-1,-1),(1,-1)$.

Solution 2.4.4. To find the singular points, we set the denominator to zero

$$2z^2+3z-2=0,$$

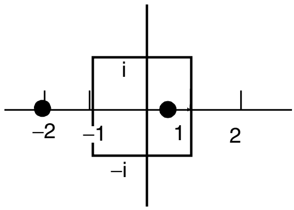
which gives the singular points at

$$z=\frac{1}{4}\left(-3\pm\sqrt{9+16}\right)=\begin{cases}\frac{1}{2},\\-2.\end{cases}$$

The denominator can be written as

$$2z^2+3z-2=2\left(z-\frac{1}{2}\right)(z+2).$$

The singular points and the contour are shown in the following figure:



Since only the singular point at $z=\frac{1}{2}$ is inside the contour, we can write the integral as

$$\oint\frac{z-1}{2z^2+3z-2}\mathrm{d}z=\oint\frac{z-1}{2(z-\frac{1}{2})(z+2)}\mathrm{d}z=\oint\frac{f(z)}{(z-\frac{1}{2})}\mathrm{d}z=2\pi\mathrm{i}\,f\left(\frac{1}{2}\right),$$

where

$$f(z)=\frac{z-1}{2(z+2)},\qquad f\left(\frac{1}{2}\right)=-\frac{1}{10}.$$

Therefore

$$\oint\frac{z-1}{2z^2+3z-2}\mathrm{d}z=-\frac{1}{5}\pi\mathrm{i}.$$

Several important theorems can be easily proved by Cauchy's integral formula and its derivatives.

Gauss' Mean Value Theorem

If $f(z)$ is analytic inside and on a circle C with center at z_0, then the mean value of $f(z)$ on C is $f(z_0)$.

This theorem follows directly from the Cauchy's integral formula:

$$f(z_0)=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{f(z)}{z-z_0}\mathrm{d}z.$$

Let the circle C be $|z - z_0| = r$, thus

$$z = z_0 + r\,\mathrm{e}^{\mathrm{i}\theta}, \quad \text{and} \quad \mathrm{d}z = \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta.$$

Therefore

$$\begin{aligned} f(z_0) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z &= \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})}{r\,\mathrm{e}^{\mathrm{i}\theta}}\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta \\ &= \frac{1}{2\pi}\oint_C f\left(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta, \end{aligned}$$

which is the mean value of $f(z)$ on C.

Liouville's Theorem

If $f(z)$ is analytic in the entire complex plane and $|f(z)|$ is bounded for all values of z, then $f(z)$ is a constant.

To prove this theorem, we start with

$$f'(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z')}{(z' - z)^2}\mathrm{d}z'.$$

The condition that $|f(z)|$ is bounded tells us that a nonnegative constant M exists such that $|f(z)| \leq M$ for all z. If we take C to be the circle $|z' - z| = R$, then

$$\begin{aligned} |f'(z)| &\leq \left|\frac{1}{2\pi\mathrm{i}}\right|\oint_C \frac{|f(z')|}{|(z' - z)^2|}\,|\mathrm{d}z'| \\ &\leq \frac{1}{2\pi}\frac{1}{R^2}M2\pi R = \frac{M}{R}. \end{aligned}$$

Since $f(z')$ is analytic everywhere, we may take R as large as we like. It is clear that $\frac{M}{R} \to 0$, as $R \to \infty$. Therefore $|f'(z)| = 0$, which implies that $f'(z) = 0$ for all z, so $f(z)$ is a constant.

Fundamental Theorem of Algebra

The following theorem is now known as the fundamental theorem of algebra. In the last chapter we mentioned that this theorem is of critical importance in our number system.

Every polynomial equation

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z^n = 0$$

of degree one or greater has at least one root.

To prove this theorem, let us first assume the contrary, namely that $P_n(z) \neq 0$ for any z. Then the function

$$f(z) = \frac{1}{P_n(z)}$$

is analytic everywhere. Since nowhere will $f(z)$ go to infinity and $f(z) \to 0$ as $z \to \infty$, so $|f(z)|$ is bounded for all z. By Liouville's theorem we conclude that $f(z)$ must be a constant, and hence $P_n(z)$ must be a constant. This is a contradiction, since $P_n(z)$ is given as a polynomial of z. Therefore, $P_n(z) = 0$ must have at least one root.

It follows from this theorem that $P_n(z) = 0$ has exactly n roots. Since $P_n(z) = 0$ has at least one root, let us denote that root $\dot{z}_1$. Thus

$$P_n(z) = (z - z_1)Q_{n-1}(z),$$

where $Q_{n-1}(z)$ is a polynomial of degree $n-1$. By the same argument, we conclude that $Q_{n-1}(z)$ must have at least one root, which we denote it as z_2. Repeating this procedure n times we find

$$P_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n) = 0.$$

Hence $P_n(z) = 0$ has exactly n roots.

Exercises

1. Show that the real and the imaginary parts of the following functions $f(z)$ satisfy the Cauchy–Reimann conditions
   (a) z^2, (b) e^z, (c) $\frac{1}{z+2}$.
2. Show that both the real part $u(x,y)$ and the imaginary part $v(x,y)$ of the analytic function $\mathrm{e}^z = u(x,y) + \mathrm{i}v(x,y)$ satisfy the Laplace equation
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$
3. Show that the derivative of $\frac{1}{z+2}$ calculated in the following three different ways gives the same result:
   (a) Let $\Delta y = 0$, so that $\Delta z \to 0$ parallel to the x-axis. In this case
$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}.$$
   (b) Let $\Delta x = 0$, so that $\Delta z \to 0$ parallel to the y-axis. In this case
$$f'(z) = \frac{\partial u}{\mathrm{i}\partial y} + \frac{\partial v}{\partial y}.$$
   (c) Use the same rule as if z were a real variable. That is
$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z}.$$

4. Let $z^2 = u(x,y) + \mathrm{i}v(x,y)$, find the point of intersection of $u(x,y) = 1$ and $v(x,y) = 2$. Show that at the point of intersection the curve $u(x,y) = 1$ is perpendicular to $v(x,y) = 2$.

5. Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be an analytic function. If $u(x,y)$ is given by the following function:
$$(\mathrm{a}) x^2 - y^2; \qquad (b) \quad \mathrm{e}^y \sin x,$$
show that they satisfy the Laplace equation. Find the corresponding conjugate harmonic function $v(x,y)$. Express $f(z)$ as a function of z only.
Ans. (a) $v(x,y) = 2xy + c, \quad f(z) = z^2 + c$. (b) $v(x,y) = \mathrm{e}^y \cos x + c, \quad f(z) = \mathrm{i}\mathrm{e}^{-\mathrm{i}z} + c$.

6. In which quadrants of the complex plane is $f(z) = |x| - \mathrm{i}|y|$ an analytic function?
Hint: In first quadrant, $x > 0$, so $\frac{\partial u}{\partial x} = \frac{\partial |x|}{\partial x} = \frac{\partial x}{\partial x} = 1$, in the second quadrant, $x < 0$, so $\frac{\partial u}{\partial x} = \frac{\partial |x|}{\partial x} = \frac{\partial (-x)}{\partial x} = -1$, and so on.
Ans. $f(z)$ is analytic only in the second and fourth quadrants.

7. Express the real part and the imaginary part of $(z+1)^2$ in terms of polar coordinates, that is, find $u(r,\theta)$ and $v(r,\theta)$ in the expression
$$(z+1)^2 = u(r,\theta) + \mathrm{i}v(r,\theta).$$
Show that they satisfy the Cauchy–Riemann equations in the polar form:
$$\frac{\partial u(r,\theta)}{\partial r} = \frac{1}{r}\frac{\partial v(r,\theta)}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u(r,\theta)}{\partial \theta} = -\frac{\partial v(r,\theta)}{\partial r}.$$

8. Show that when an analytic function is expressed in terms of polar coordinates, both its real part and its imaginary part satisfy Laplace's equation in polar coordinates
$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

9. To show that line integral are, in general, dependent on the path of integration, evaluate
$$\int_{-1}^{\mathrm{i}} |z|^2 \, \mathrm{d}z$$
(a) along the straight line from the initial point -1 to the final point i, (b) along the arc of the unit circle $|z| = 1$ traversed in the clockwise direction from the initial point -1 to the final point i.
Hint: (a) Parameterize the line segment by $z = -1 + (1+\mathrm{i})t, \quad 0 \le t \le 1$. (b) Parameterize the arc by $z = \mathrm{e}^{\mathrm{i}\theta}, \pi \ge \theta \ge \pi/2$.
Ans. (a) $2(1+\mathrm{i})/3, (b) 1 + \mathrm{i}$.

10. To verify that the line integral of an analytic function is independent of the path, evaluate

$$\int_0^{3+\mathrm{i}} z^2 \mathrm{d}z$$

(a) along the line $y = x/3$, (b) along the real axis to 3 and then vertically to $3 + \mathrm{i}$, (c) along the imaginary axis to i and then horizontally to $3 + \mathrm{i}$.

Ans. (a) $6 + \frac{26}{3}\mathrm{i}$, (b) $6 + \frac{26}{3}\mathrm{i}$, (c) $6 + \frac{26}{3}\mathrm{i}$.

11. Verify the Green's lemma

$$\oint [A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y] = \iint\limits_R \left[\frac{\partial B(x,y)}{\partial x} - \frac{\partial A(x,y)}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y$$

for the integral

$$\oint [(x^2 + y)\mathrm{d}x - xy^2\mathrm{d}y]$$

taken around the boundary of the square with vertices at (0,0), (1,0), (0,1), (1,1).

12. Verify the Green's lemma for the integral

$$\oint [(x - y)\mathrm{d}x + (x + y)\mathrm{d}y]$$

taken around the boundary of the area in the first quadrant between the curve $y = x^2$ and $y^2 = x$.

13. Evaluate

$$\int_0^{3+\mathrm{i}} z^2 \mathrm{d}z$$

with fundamental theorem of calculus. That is,

$$\text{if} \quad \frac{\mathrm{d}F(z)}{\mathrm{d}z} = f(z), \quad \text{then} \quad \int_A^B f(z)\,\mathrm{d}z = F(B) - F(A),$$

provided $f(z)$ is analytic in a region between A and B.

Ans. $6 + \frac{26}{3}\mathrm{i}$.

14. What is the value of

$$\oint_C \frac{3z^2 + 7z + 1}{z + 1} \mathrm{d}z$$

(a) if C is the circle $|z + 1| = 1$? (b) if C is the circle $|z + \mathrm{i}| = 1$? (c) if C is the ellipse $x^2 + 2y^2 = 8$?

Ans. (a) $-6\pi\mathrm{i}$, (b) 0, (c) $-6\pi\mathrm{i}$.

15. What is the value of

$$\oint\limits_C \frac{z+4}{z^2+2z+5}\mathrm{d}z$$

(a) if C is the circle $|z|=1$? (b) if C is the circle $|z+1-\mathrm{i}|=2$? (c) if C is the circle $|z+1+\mathrm{i}|=2$?

Ans. (a) 0, (b) $\frac{1}{2}(3+2\mathrm{i})\,\pi$, (c) $\frac{1}{2}(-3+2\mathrm{i})\,\pi$.

16. What is the value of

$$\oint\limits_C \frac{\mathrm{e}^z}{(z+1)^2}\mathrm{d}z$$

around the circle $|z-1|=3$?

Ans. $2\pi\mathrm{i}\mathrm{e}^{-1}$.

17. What is the value of

$$\oint\limits_C \frac{z+1}{z^3-2z^2}\mathrm{d}z$$

(a) If C is the circle $|z|=1$? (b) If C is the circle $|z-2-\mathrm{i}|=2$? (c) If C is the circle $|z-1-2\mathrm{i}|=2$?

Ans. (a) $-\frac{3}{2}\pi\mathrm{i}$, (b) $\frac{3}{2}\pi\mathrm{i}$, (c) 0.

18. Find the value of the closed loop integral

$$\oint \frac{z^3+\sin z}{(z-\mathrm{i})^3}\mathrm{d}z$$

taken around the boundary of the triangle with vertices at $\pm 2,\ 2\mathrm{i}$.

Ans. $\pi\left(\mathrm{e}-\mathrm{e}^{-1}\right)/2-6\pi$.

19. What is the value of

$$\oint\limits_C \frac{\tan z}{z^2}\mathrm{d}z$$

if C is the circle $|z|=1$?

Ans. $2\pi\mathrm{i}$.

20. What is the value of

$$\oint\limits_C \frac{\ln z}{(z-2)^2}\mathrm{d}z$$

if C is the circle $|z-3|=2$?

Ans. $\pi\mathrm{i}$.

3

Complex Series and Theory of Residues

Series expansions are ubiquitous in science and engineering. In the theory of complex functions, series expansions play a crucial role because they are the basis for deriving and using the theory of residues, which provide a powerful method for calculating both complex contour integrals and some difficult integrals of real variable. Before the formal development, we will first review a basic geometric series.

3.1 A Basic Geometric Series

Let

$$S = 1 + z + z^2 + z^3 + \cdots + z^n. \tag{3.1}$$

Multiplying by z,

$$zS = z + z^2 + z^3 + \cdots + z^n + z^{n+1}$$

and subtracting the two series

$$(1 - z)S = 1 - z^{n+1}$$

we get

$$S = \frac{1 - z^{n+1}}{1 - z}.$$

Now if $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$. Thus, if n goes to infinity,

$$S = \frac{1}{1 - z},$$

and it follows from (3.1) that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k=0}^{\infty} z^k. \tag{3.2}$$

Clearly it will diverge for $|z| \geq 1$. It is important to remember that this series converges only for $|z| < 1$. Under this condition, the following alternative series is also convergent:

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{k=0}^{\infty} (-z)^k = 1 - z + z^2 - z^3 + \cdots. \tag{3.3}$$

3.2 Taylor Series

Taylor series is perhaps the most familiar series in real variables. Taylor series in complex variable is even more interesting.

3.2.1 The Complex Taylor Series

In many applications of complex variables, we wish to expand an analytic function $f(z)$ into a series around a particular point $z = z_0$. We will show that if $f(z)$ is analytic in the neigborhood of z_0 including the point at $z = z_0$, then $f(z)$ can be represented as a series of positive powers of $(z - z_0)$.

First let us recall

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z} \, \mathrm{d}t, \tag{3.4}$$

where t is the integration variable and it is on the enclosed contour C, inside which $f(z)$ is analytic. The quantity $(z - z_0)$ can be introduced into the integral through the identity

$$\frac{1}{t-z} = \frac{1}{(t-z_0) + (z_0 - z)} = \frac{1}{(t-z_0)\left(1 - \frac{z-z_0}{t-z_0}\right)}.$$

If

$$\left| \frac{z-z_0}{t-z_0} \right| < 1, \tag{3.5}$$

then by the basic geometric series (3.2)

$$\frac{1}{1 - \frac{z-z_0}{t-z_0}} = 1 + \left(\frac{z-z_0}{t-z_0}\right) + \left(\frac{z-z_0}{t-z_0}\right)^2 + \left(\frac{z-z_0}{t-z_0}\right)^3 + \cdots. \tag{3.6}$$

Therefore (3.4) can be written as

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z}\, \mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z_0} \left[1 + \left(\frac{z-z_0}{t-z_0} \right) + \left(\frac{z-z_0}{t-z_0} \right)^2 + \left(\frac{z-z_0}{t-z_0} \right)^3 + \cdots \right] \mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z_0}\, \mathrm{d}t + \left[\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^2}\, \mathrm{d}t \right] (z-z_0) \\ &\quad + \left[\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^3}\, \mathrm{d}t \right] (z-z_0)^2 \\ &\quad + \left[\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^4}\, \mathrm{d}t \right] (z-z_0)^3 + \cdots . \end{aligned} \tag{3.7}$$

According to Cauchy's integral formula and its derivatives

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^{n+1}}\, \mathrm{d}t$$

the earlier equation (3.7) becomes

$$\begin{aligned} f(z) &= \sum_{n=0} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \\ &= f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots . \end{aligned} \tag{3.8}$$

This is the well-known Taylor series.

3.2.2 Convergence of Taylor Series

To discuss the convergence of the Taylor series, let us first recall the definition of singular points.

Singularity

If $f(z)$ is analytic at all points in the neighborhood of z_s but is not differentiable at z_s, then z_s is called a singular point. We also say that $f(z)$ has a singularity at $z = z_s$. For example:

$\frac{1}{z^2+1}$ has singularities at $z = \mathrm{i},\ -\mathrm{i}$.

$\tan z = \frac{\sin z}{\cos z}$ has singularities at $z = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, \ldots$

$\frac{1+2z}{z^2-5z+6}$ has singularities at $z = 2,\ 3$.

$\frac{1}{\mathrm{e}^z+1}$ has singularities at $z = \pm\mathrm{i}\pi,\ \pm\mathrm{i}3\pi,\ \pm\mathrm{i}5\pi, \ldots$.

Radius of Convergence

The Cauchy integral formula of (3.4) is, of course, valid for all z inside the Contour C, if $f(t)$ is analytic in and on C. However, in developing the Taylor series around $z = z_0$, we have used (3.6), which is true only if the condition of (3.5) is satisfied. This means $|z - z_0|$ must be less than $|t - z_0|$. Since t is on the contour C as shown in Fig. 3.1, the distance $|t - z_0|$ is changing as t is moving around C. With the contour shown in the figure, the smallest $|t - z_0|$ is $|s - z_0|$ where s is the point on C closest to z_0. For $|z - z_0|$ to be less than all possible $|t - z_0|$, $|z - z_0|$ must be less than $|s - z_0|$. This means the Taylor series of (3.8) is valid only for those points of z which are inside the circle centered at z_0, with a radius $R = |s - z_0|$.

If $f(z)$ is analytic everywhere, we can draw the contour C as large as we want. Therefore the Taylor series is convergent in the entire complex plane. However, if $f(z)$ has a singular point at $z = s$, then the contour must be so drawn in such way that the point $z = s$ is outside of C. In Fig. 3.1, the contour C can be infinitesimally close to s, but s must not be on or inside C. For such a case the largest possible radius of convergence is $|s - z_0|$. Therefore the radius of convergence of a Taylor series is equal to the distance between its expansion center and the nearest singular point.

The discussion earlier applies equally well to a circular region about the origin, $z_0 = 0$. The Taylor series about the origin

$$f(z) = f(0) + f'(0)\, z + \frac{f''(0)}{2!} z^2 + \cdots$$

is called the Maclaurin series.

Even in the expansion of a function of a real variable, the radius of convergence is equally important. To illustrate, consider

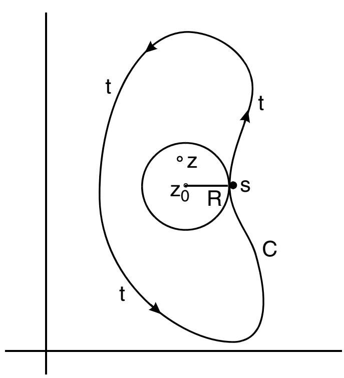


Fig. 3.1. Radius of convergence of the Taylor series. The expansion center is at z_0. The singular point at s limits the region of convergence within the interior of the circle of radius R

$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots .$$

This series converges throughout the interior of the largest circle around the origin in which $f(z)$ is analytic. Now, $f(z)$ has two singular points at $z = \pm \mathrm{i}$, and even though one may be concerned solely with real values of z, for which $1/(1+x^2)$ is everywhere infinitely differentiable with respect to x, these singularities in the complex plane set an inescapable limit to the interval of convergence on the x axis. Since the distance between the expansion center at $z = 0$ and the nearest singular point, i or $-\mathrm{i}$ is $|\mathrm{i} - 0| = 1$, the radius of convergence is equal to one. The series is convergent only inside the circle of radius 1, centered at origin. Thus the interval of convergence on the x axis is between $x = \pm 1$. In other words, the Maclaurin series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

is valid only for $-1 < x < 1$, although $1/(1+x^2)$ and its derivatives of all orders are well defined along the real axis x. Now if we expand the real function $1/(1+x^2)$ into Taylor series around $x = x_0$, then the radius of convergence is equal to $|\mathrm{i} - x_0| = \sqrt{1+x_0^2}$. This means that this series will converge only in the interval between $x = x_0 - \sqrt{1+x_0^2}$ and $x = x_0 + \sqrt{1+x_0^2}$.

3.2.3 Analytic Continuation

If we know the values of an analytic function in some small region around z_0, we can use the Taylor expansion about z_0 to find the values of the function in a larger region. Although the Taylor expansion is valid only inside the circle of radius of convergence which is determined by the location of the nearest singular point, a chain of Taylor expansions can be used to determine the function throughout the entire complex plane except at the singular points of the function. This process is illustrated in Fig. 3.2.

Suppose we know the values around z_0 and the singular point nearest to z_0 is s_0. The Taylor expansion about z_0 holds within a circular region of radius $|z_0 - s_0|$. Since the Taylor expansion gives the values of the function and all its derivatives at every point in this circle, we can use any point in this circle as the new expansion center. For example, we may expand another Taylor series about z_1 as shown in Fig. 3.2. We can do this because $f^n(z_1)$ is known for all n from the first Taylor expansion about z_0. The radius of convergence of this second Taylor series is determined by the distance from z_1 to the nearest singular point s_1. Continuing this way, as indicated in Fig. 3.2, we can cover the whole complex plane except at the singular points $s_0, s_1, s_2, \ldots$. In other words, the analytic function everywhere can be constructed from the knowledge of the function in a small region. This process is called analytic continuation.

An immediate consequence of analytic continuation is the so called identity theorem. It states that if $f(z)$ and $g(z)$ are analytic and $f(z) = g(z)$ along

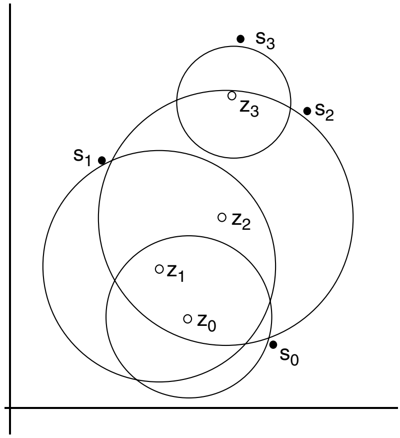


Fig. 3.2. Analytic Continuation. A series of Taylor expansions which analytically continue a function originally known in the region around z_0. The first expansion about z_0 is valid only inside the circle of radius $|z_0 - s_0|$, where s_0 is the singular point nearest to z_0. The next Taylor expansion is around z_1 which is inside the first circle. The second Taylor expansion is limited by the singular point s_1, and so on

a curve L in a region D, then $f(z) = g(z)$ throughout D. We can show this by considering the analytic function $h(z) = f(z) - g(z)$. If we can show that $h(z)$ is identically zero throughout the region, then the theorem is proved.

If we choose a point $z = z_0$ on L, then we can expand $h(z)$ in a Taylor series about z_0,

$$h(z) = h(z_0) + h'(z_0)(z - z_0) + \frac{1}{2!}h''(z_0)(z - z_0)^2 + \cdots ,$$

which will converge inside the some circle that extends as far as the nearest point of the boundary of D. But since z_0 is on L, $h(z_0) = 0$. Furthermore, the derivatives of h must also be zero if z is approaching z_0 along L. Since $h(z)$ is analytic, its derivatives are independent on the way how z is approaching z_0, this means

$$h'(z_0) = h''(z_0) = \cdots = 0.$$

Therefore, $h(z) = 0$ inside the circle. We may now expand about a new point, which can lie anywhere inside the circle. Thus by analytic continuation, we may show that $h(z) = 0$ throughout the region D.

3.2.4 Uniqueness of Taylor Series

If there are constants $a_n \ (n = 0,\ 1,\ 2, \ldots)$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is convergent for all points z interior to some circle centered at z_0, then this power series must be the Taylor series, regardless of how those constants are obtained. This is quite easy to show, since

$$\begin{aligned} f(z) &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots, \\ f'(z) &= a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + 4a_4(z-z_0)^3 + \cdots, \\ f''(z) &= 2a_2 + 3\cdot 2a_3(z-z_0) + 4\cdot 3(z-z_0)^2 + \cdots, \end{aligned}$$

clearly

$$f(z_0) = a_0, \quad f'(z_0) = a_1, \quad f''(z_0) = 2a_2, \quad f'''(z_0) = 3\cdot 2a_3, \quad \ldots.$$

It follows that:

$$a_n = \frac{1}{n!} f^{(n)}(z_0),$$

which are the Taylor coefficients. Thus, Taylor series is unique. Thus no matter how the power series is obtained, if it is convergent in some circular region, it is the Taylor series. The following examples illustrate some of the techniques of expanding a function into its Taylor series.

Example 3.2.1. Find the Taylor series about the origin and its radius of convergence for

$$\text{(a)}\ \sin z, \quad \text{(b)}\ \cos z, \quad \text{(c)}\ \mathrm{e}^z.$$

Solution 3.2.1. (a) Since $f(z) = \sin z$,

$$f'(z) = \cos z, \quad f''(z) = -\sin z, \quad f'''(z) = -\cos z, \quad f^4(z) = \sin z, \quad \ldots.$$

Hence

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^4(0) = 0, \quad \ldots.$$

Thus

$$\begin{aligned} \sin z &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n \\ &= z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots. \end{aligned}$$

This series is valid for all z, since $\sin z$ is an entire function (analytic for the entire complex plane).

(b) If $f(z) = \cos z$, then

$$f'(z) = -\sin z, \quad f''(z) = -\cos z, \quad f'''(z) = \sin z, \quad f^4(z) = \cos z, \quad \ldots$$

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^4(0) = 1, \quad \ldots .$$

Therefore

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots .$$

This series is also valid for all z.

(c) For $f(z) = \mathrm{e}^z$, then

$$f^{(n)}(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\,\mathrm{e}^z = \mathrm{e}^z \text{and} f^{(n)}(0) = 1.$$

It follows:

$$\mathrm{e}^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots .$$

This series converges for all z, since e^z is an entire function.

Example 3.2.2. Find the Taylor series about the origin and its radius of convergence for

$$f(z) = \frac{\mathrm{e}^z}{\cos z}.$$

Solution 3.2.2. The singular points of the function are at the zeros of the denominator. Since $\cos\frac{\pi}{2} = 0$, the singular point nearest to the origin is at $z = \pm\frac{\pi}{2}$. Therefore the Taylor series about the origin is valid for $|z| < \frac{\pi}{2}$. We can find the constants a_n of

$$\frac{\mathrm{e}^z}{\cos z} = a_0 + a_1 z + a_2 z^2 + \cdots$$

from $f^{(n)}(0)$, but the repeated differentiations become increasingly tedious. So we take the advantage of the fact that the Taylor series for e^z and $\cos z$ are already known. Replacing e^z and $\cos z$ with their respective Taylor series in the equation

$$\mathrm{e}^z = \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)\cos z,$$

we obtain

$$1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots = \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)$$
$$\left(1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots\right).$$

Multiplying out and collecting terms, we have

$$1+z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\cdots=a_0+a_1z+\left(a_2-\frac{1}{2}a_0\right)z^2+\left(a_3-\frac{1}{2}a_1\right)z^3+\cdots.$$

Therefore

$$a_0=1,\quad a_1=1,\quad a_2-\frac{1}{2}a_0=\frac{1}{2},\quad a_3-\frac{1}{2}a_1=\frac{1}{3!},\ \ldots$$

It follows that:

$$a_2=\frac{1}{2}+\frac{1}{2}a_0=1,\quad a_3=\frac{1}{3!}+\frac{1}{2}a_1=\frac{2}{3},\quad \ldots$$

and

$$\frac{\mathrm{e}^z}{\cos z}=1+z+z^2+\frac{2}{3}z^3+\cdots,\qquad |z|<\frac{\pi}{2}.$$

Example 3.2.3. Find the Taylor series about $z=2$ for

$$\text{(a) } \frac{1}{z},\qquad \text{(b) } \frac{1}{z^2}.$$

Solution 3.2.3. (a) The function $1/z$ has a singular point at $z=0$, the distance between this point and the expansion center is 2. Therefore the Taylor series about $z=2$ is convergent for $|z-2|<2$ and has the form

$$\frac{1}{z}=a_0+a_1(z-2)+a_2(z-2)^2+\cdots.$$

We can write the function as

$$\frac{1}{z}=\frac{1}{2+(z-2)}=\frac{1}{2}\frac{1}{1+\left(\frac{z-2}{2}\right)}.$$

For $|z-2|<2$, $\left|\frac{z-2}{2}\right|<1$. Therefore we can use the geometric series (3.3) to expand

$$\frac{1}{1+\left(\frac{z-2}{2}\right)}=1-\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\cdots.$$

It follows that for $|z-2|<2$

$$\begin{aligned}\frac{1}{z}&=\frac{1}{2}\left[1-\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\left(\frac{z-2}{2}\right)^4-\cdots\right]\\&=\frac{1}{2}-\frac{1}{4}(z-2)+\frac{1}{8}(z-2)^2-\frac{1}{16}(z-2)^3+\frac{1}{32}(z-2)^4-\cdots.\end{aligned}$$

(b) Since

$$\frac{1}{z^2} = -\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{z},$$

therefore

$$\begin{aligned}\frac{1}{z^2} &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{2} - \frac{1}{4}(z-2) + \frac{1}{8}(z-2)^2 - \frac{1}{16}(z-2)^3 + \frac{1}{32}(z-2)^4 - \cdots\right) \\ &= \frac{1}{4} - \frac{1}{4}(z-2) + \frac{3}{16}(z-2)^2 - \frac{1}{8}(z-2)^3 + \cdots .\end{aligned}$$

Example 3.2.4. Find the Taylor series about the origin for

$$f(z) = \frac{1}{1+z-2z^2}.$$

Solution 3.2.4. Since $1+z-2z^2 = (1-z)(1+2z)$, the function $f(z)$ has two singular points at $z=1$ and $z=-1/2$. The Taylor series expansion about $z=0$ will be convergent for $|z| < \frac{1}{2}$. Furthermore

$$\frac{1}{1+z-2z^2} = \frac{1/3}{1-z} + \frac{2/3}{1+2z}.$$

For $|z| < \frac{1}{2}$ and $|2z| < 1$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots ,$$

$$\frac{1}{1+2z} = 1 - 2z + 4z^2 - 8z^3 + \cdots .$$

Thus

$$\begin{aligned}f(z) &= \frac{1}{3}\left(1 + z + z^2 + z^3 + \cdots\right) + \frac{2}{3}\left(1 - 2z + 4z^2 - 8z^3 + \cdots\right) \\ &= 1 - z + 3z^2 - 5z^3 + \cdots .\end{aligned}$$

Example 3.2.5. Find the Taylor series about the origin for

$$f(z) = \ln(1+z).$$

Solution 3.2.5. First note that

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln(1+z) = \frac{1}{1+z},$$

and

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots,$$

so

$$\mathrm{d}\ln(1+z) = \left(1 - z + z^2 - z^3 + \cdots\right)\mathrm{d}z.$$

Integrating both sides, we have

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots + k.$$

The integration constant $k = 0$, since at $z = 0$, $\ln(1) = 0$. Therefore

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots.$$

This series converges for $|z| < 1$, since at $z = -1$, $f(z)$ is singular.

3.3 Laurent Series

In many applications, it is necessary to expand functions around points at which, or in the neighborhood of which, the functions are not analytic. The method of Taylor series is obviously inapplicable in such cases. A new type of series known as Laurent expansion is required. This series furnishes us with a representation which is valid in the annular ring bounded by two concentric circles, provided that the function being expanded is analytic everywhere on and between the two circles.

Consider the annulus bounded by two circles of C_0 and C_i with a common center z_0 as shown in Fig. 3.3a. The function $f(z)$ is analytic inside the annular region; however, there may be singular points inside the smaller circle

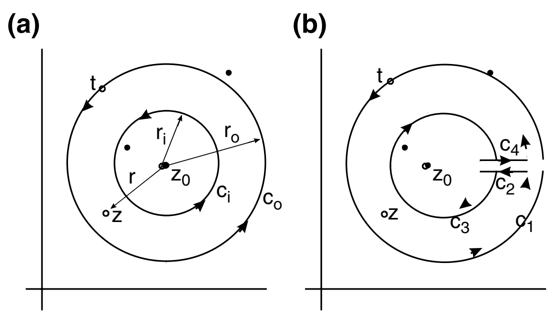


Fig. 3.3. Annular region between two *circles* where the function is analytic and the Laurent series is valid. Inside the *inner circle* and outside the *outer circle*, the function may have singular points

or outside the larger circle. We can apply the Cauchy's integral formula to the region which is cut up as shown in Fig. 3.3b. The region is now simply connected and is bounded by the curve $C' = C_1 + C_2 + C_3 + C_4$. Cauchy's integral formula is then

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_{C'} \frac{f(t)}{t-z} \mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \left(\int_{C_1} \frac{f(t)}{t-z} \mathrm{d}t + \int_{C_2} \frac{f(t)}{t-z} \mathrm{d}t + \int_{C_3} \frac{f(t)}{t-z} \mathrm{d}t + \int_{C_4} \frac{f(t)}{t-z} \mathrm{d}t \right), \end{aligned}$$

where t is on C' and z is a point inside C'. Now let the gap between C_2 and C_4 shrink to zero, then the integrals along C_2 and C_4 will cancel each other, since they are oriented in the opposite directions, if $f(z)$ is single valued. Furthermore, the contour C_1 becomes C_0 and the contour C_3 is identical to C_i turning the opposite direction. Therefore

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{f(t)}{t-z} \mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \oint_{C_i} \frac{f(t)}{t-z} \mathrm{d}t, \tag{3.9}$$

where C_0 and C_i are both traversed in the counterclockwise direction. The negative sign results because the direction of integration was reversed on C_i.

We can introduce z_0, the common center of C_0 and C_i, as the expansion center. For the first integral in (3.9) with t on C_0, we can write

$$\begin{aligned} \frac{1}{t-z} &= \frac{1}{t-z_0+z_0-z} = \frac{1}{(t-z_0)-(z-z_0)} \\ &= \frac{1}{(t-z_0)\left(1-\frac{z-z_0}{t-z_0}\right)}. \end{aligned} \tag{3.10}$$

Since t is on C_0 and z is inside C_0, as shown in Fig. 3.3a

$$\left| \frac{z-z_0}{t-z_0} \right| = \frac{r}{r_o} < 1,$$

so we can expand $\left(1-\frac{z-z_0}{t-z_0}\right)^{-1}$ with the geometric series of (3.2), and (3.10) becomes

$$\begin{aligned} \frac{1}{t-z} = \frac{1}{t-z_0} \Bigg[1 &+ \frac{z-z_0}{t-z_0} + \left(\frac{z-z_0}{t-z_0}\right) + \left(\frac{z-z_0}{t-z_0}\right)^2 \\ &+ \left(\frac{z-z_0}{t-z_0}\right)^3 + \cdots \Bigg] \quad \text{for } t \text{ on } C_0. \end{aligned} \tag{3.11}$$

For the second integral with t on C_i and z is between C_0 and C_i, we can write

$$\frac{1}{t-z} = -\frac{1}{z-t} = -\frac{1}{z-z_0+z_0-t}$$
$$= -\frac{1}{(z-z_0)-(t-z_0)} = -\frac{1}{(z-z_0)\left[1-\frac{t-z_0}{z-z_0}\right]}.$$

Since

$$\left|\frac{t-z_0}{z-z_0}\right| = \frac{r_i}{r} < 1$$

as shown in Fig. 3.3a, we can again expand

$$\left(1-\frac{t-z_0}{z-z_0}\right)^{-1}$$

with the geometric series, and write

$$\frac{1}{t-z} = -\frac{1}{z-z_0}\left[1+\frac{t-z_0}{z-z_0}+\left(\frac{t-z_0}{z-z_0}\right)^2+\left(\frac{t-z_0}{z-z_0}\right)^3+\cdots\right] \quad \text{for } t \text{ on } C_i. \tag{3.12}$$

Putting (3.11) and (3.12) into (3.9), we have

$$f(z) = I_{C_0} + I_{C_i},$$

where

$$I_{C_0} = \frac{1}{2\pi\mathrm{i}}\oint_{C_0}\frac{f(t)}{t-z_0}\left[1+\frac{z-z_0}{t-z_0}+\left(\frac{z-z_0}{t-z_0}\right)^2+\left(\frac{z-z_0}{t-z_0}\right)^3+\cdots\right]\mathrm{d}t$$
$$= \frac{1}{2\pi\mathrm{i}}\oint_{C_0}\frac{f(t)}{t-z_0}\,\mathrm{d}t + \left(\frac{1}{2\pi\mathrm{i}}\oint_{C_0}\frac{f(t)}{(t-z_0)^2}\,\mathrm{d}t\right)(z-z_0)$$
$$+\left(\frac{1}{2\pi\mathrm{i}}\oint_{C_0}\frac{f(t)}{(t-z_0)^3}\,\mathrm{d}t\right)(z-z_0)^2+\left(\frac{1}{2\pi\mathrm{i}}\oint_{C_0}\frac{f(t)}{(t-z_0)^4}\,\mathrm{d}t\right)(z-z_0)^3+\cdots$$

and

$$I_{C_i} = \frac{1}{2\pi\mathrm{i}}\oint_{C_i}\frac{f(t)}{z-z_0}\left[1+\frac{t-z_0}{z-z_0}+\left(\frac{t-z_0}{z-z_0}\right)^2+\left(\frac{t-z_0}{z-z_0}\right)^3+\cdots\right]\mathrm{d}t$$
$$= \left(\frac{1}{2\pi\mathrm{i}}\oint_{C_i}f(t)\mathrm{d}t\right)\frac{1}{z-z_0} + \left(\frac{1}{2\pi\mathrm{i}}\oint_{C_i}f(t)(t-z_0)\mathrm{d}t\right)\frac{1}{(z-z_0)^2}$$
$$+\left(\frac{1}{2\pi\mathrm{i}}\oint_{C_i}f(t)(t-z_0)^2\mathrm{d}t\right)\frac{1}{(z-z_0)^3}+\cdots.$$

Therefore, in the region between C_i and C_0, $f(z)$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{k=1}^{\infty} b_n \frac{1}{(z - z_0)^k},$$

where

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{f(t)}{(t - z_0)^{n+1}} \, \mathrm{d}t, \qquad b_k = \frac{1}{2\pi \mathrm{i}} \oint_{C_i} f(t)\,(t - z_0)^{k-1} \, \mathrm{d}t.$$

Because of the principle of deformation of contours, we can replace both C_i and C_0 by a closed contour C between C_i and C_0 without changing the values of the integrals. Thus we can write this series as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

with

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C} \frac{f(t)}{(t - z_0)^{n+1}} \, \mathrm{d}t. \tag{3.13}$$

This expansion is known as the Laurent series which contains both negative and positive powers of $(z - z_0)$.

It should be noted that the coefficients of positive powers of $(z - z_0)$ cannot be replaced by the derivative expressions, since $f(z)$ is not analytic inside C. However, if there is no singular point inside C_i, then these coefficients can indeed be replaced by $f^{(n)}(z_0)/n!$, at the same time the coefficients of the negative powers of $(z - z_0)$ are identically equal to zero by the Cauchy theorem, since $f(t)(t - z_0)^{-n-1}$ for $n \leq -1$ are analytic inside C. In such a case, the Laurent expansion reduces to the Taylor expansion.

3.3.1 Uniqueness of Laurent Series

Just as Taylor series, Laurent series is unique. If a series

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges to $f(z)$ at all points in some annular domain about z_0, then regardless how the constants a_n are obtained, the series is the Laurent expansion for $f(z)$ in powers of $(z - z_0)$ for that domain. This statement is proved if we can show that

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C} \frac{f(t)}{(t - z_0)^{n+1}} \, \mathrm{d}t. \tag{3.14}$$

We now show that this is indeed the case. Let $g_k(t)$ be defined as

$$g_k(t) = \frac{1}{2\pi \mathrm{i}} \frac{1}{(t-z_0)^{k+1}},$$

where k is an integer, either positive or negative, or zero. Furthermore let C be a circle inside the annulus centered at z_0 and taken in the counterclockwise direction, so

$$\oint_C g_k(t) f(t)\,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^{k+1}}\,\mathrm{d}t. \tag{3.15}$$

Now if $f(t)$ is expressible as

$$f(t) = \sum_{n=-\infty}^{\infty} a_n (t-z_0)^n,$$

then

$$\begin{aligned}\oint_C g_k(t) f(t)\,\mathrm{d}t &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{1}{(t-z_0)^{k+1}} \left(\sum_{n=-\infty}^{\infty} a_n (t-z_0)^n \right) \mathrm{d}t \\ &= \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi \mathrm{i}} \oint_C \frac{1}{(t-z_0)^{k-n+1}}\,\mathrm{d}t.\end{aligned}$$

The last integral can be easily evaluated by setting $t-z_0 = r\mathrm{e}^{\mathrm{i}\theta}$, so $\mathrm{d}t = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$ and

$$\begin{aligned}\oint_C \frac{1}{(t-z_0)^{k-n+1}}\,\mathrm{d}t &= \int_0^{2\pi} \frac{\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{r^{k-n+1}\mathrm{e}^{\mathrm{i}(k-n+1)\theta}}\,\mathrm{d}\theta \\ &= \frac{\mathrm{i}}{r^{k-n}} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-k)}\mathrm{d}\theta = 2\pi\mathrm{i}\delta_{nk} = \begin{cases} 0 & n \neq k, \\ 2\pi \mathrm{i} & n = k. \end{cases}\end{aligned} \tag{3.16}$$

Thus

$$\oint_C g_k(t) f(t)\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi \mathrm{i}} 2\pi\mathrm{i}\delta_{nk} = a_k. \tag{3.17}$$

It follows from (3.15) that:

$$a_k = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-t_0)^{k+1}}\,\mathrm{d}t. \tag{3.18}$$

Since k is an arbitrary integer, (3.14) must hold.

Thus no matter how the expansion is obtained, as long as it is valid in the specified annular domain, it is the Laurent series. This enables us to determine the Laurent coefficients with elementary techniques, as illustrated in the following examples. The integral representations of the Laurent coefficients (3.18) are important, not as means of finding the coefficients, but as means of using the coefficients to evaluate these integrals. We will elaborate this aspect of the Laurent series in following sections on the theory of residues.

Example 3.3.1. Find the Laurent series about $z = 0$ for the function

$$f(z) = \mathrm{e}^{1/z}.$$

Solution 3.3.1. Since $f(z)$ is analytic for all z, except for $z = 0$, the expansion of $f(z)$ about $z = 0$ will be a Laurent series valid in the annulus $0 < |z| < \infty$. To obtain the expansion, let $1/z = t$, and note

$$\mathrm{e}^{t} = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots .$$

Therefore

$$\mathrm{e}^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots .$$

Example 3.3.2. Find all possible Laurent expansions about $z = 0$ of

$$f(z) = \frac{1+2z^2}{z^3+z^5},$$

and specify the regions in which they are valid.

Solution 3.3.2. By setting the denominator to zero $z^3+z^5 = z^3\left(1+z^2\right) = 0$, We get three singular points, $z = 0$, and $z = \pm\mathrm{i}$. They are shown in Fig. 3.4. Therefore we can expand the function about $z = 0$ in two different Laurent series, one is valid for the region $0 < |z| < 1$ as shown in (a), the other is valid in the region $|z| > 1$ as shown in (b).

The function can be written as

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1+2z^2}{z^3(1+z^2)} = \frac{2(1+z^2)-1}{z^3(1+z^2)} = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right).$$

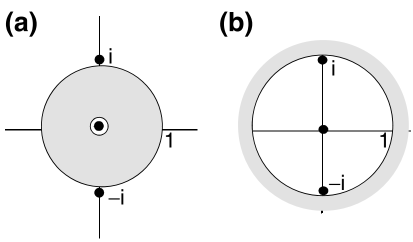


Fig. 3.4. If the function has three singular points at $z = 0$ and $z = \pm\mathrm{i}$, then the function can be expanded into two different Laurent series about $z = 0$. (**a**) One series is valid in the region $0 < |z| < 1$, (**b**) the other series is valid in the region $1 < |z|$

In the case of (a), $|z| < 1$, so is $\left|z^2\right| < 1$. We can use the geometric series to express

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots .$$

Therefore

$$\begin{aligned} f(z) &= \frac{1}{z^3}\left(2 - \left[1 - z^2 + z^4 - z^6 + \cdots\right]\right) \\ &= \frac{1}{z^3} + \frac{1}{z} - z + z^3 + \cdots \quad \text{for} \quad 0 < |z| < 1. \end{aligned}$$

In the case of (b), $\left|z^2\right| > 1$, we first write

$$\frac{1}{1+z^2} = \frac{1}{z^2\left(1+\frac{1}{z^2}\right)}.$$

Since $\left|\frac{1}{z^2}\right| < 1$, again we can use the geometric series

$$\frac{1}{\left(1+\frac{1}{z^2}\right)} = 1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots .$$

Thus

$$\begin{aligned} f(z) &= \frac{1}{z^3}\left(2 - \frac{1}{z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right]\right) \\ &= \frac{2}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} + \cdots, \quad \text{for} \quad |z| > 1. \end{aligned}$$

Example 3.3.3. Find the Laurent series expansion of

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

valid in each of the shaded regions shown in Fig. 3.5.

Solution 3.3.3. First we note that $z^2 - 3z + 2 = (z-2)(z-1)$, so the function has two singular points at $z = 2$ and $z = 1$. Taking the partial fraction, we have

$$\begin{aligned} f(z) &= \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)(z-1)} \\ &= \frac{1}{z-2} - \frac{1}{z-1}. \end{aligned}$$

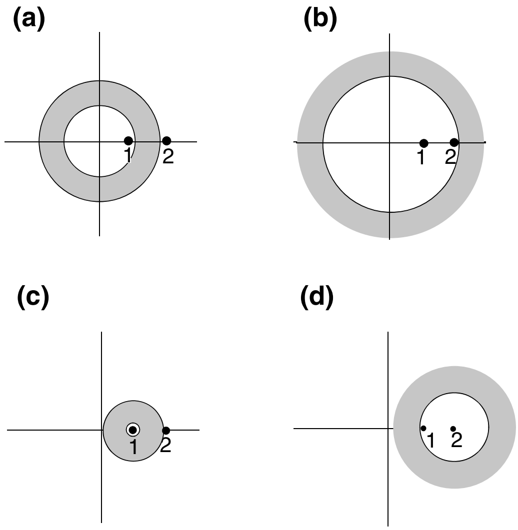


Fig. 3.5. The function with two singular points at $z = 1$ and $z = 2$ can be expanded into different Laurent series in different regions: (**a**) expanded about $z = 0$ valid in the region $1 < |z| < 2$, (**b**) expanded about $z = 0$ valid in the region $2 < |z|$, (**c**) expanded about $z = 1$ valid in the region $0 < |z - 1| < 1$, (**d**) expanded $z = 2$ valid in the region $1 < |z - 2|$

(a) In this case, we have to expand around $z = 0$, so we are seeking a series in the form of

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

The values of $|z|$ between the two circles are such that $1 < |z| < 2$. To make use of the basic geometric series, we write

$$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})}.$$

Since $|z/2| < 1$, so

$$\begin{aligned}\frac{1}{z-2} &= -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right] \\ &= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots.\end{aligned}$$

As for the second fraction, we note that $|z| > 1$, so $|1/z| < 1$. Therefore we write

$$
\begin{aligned}
-\frac{1}{z-1} &= -\frac{1}{z(1-\frac{1}{z})} \\
&= -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right] \\
&= -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots .
\end{aligned}
$$

Thus the Laurent series in region (a) is

$$
f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots .
$$

(b) Again the expansion center is at the origin, but in region (b) $|z| > 2$. So we expand the first fraction as

$$
\begin{aligned}
\frac{1}{z-2} &= \frac{1}{z\left(1-\frac{2}{z}\right)} \\
&= \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right] \\
&= \frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\cdots .
\end{aligned}
$$

Note that the expansion of the second fraction we worked out in part (a) is still valid in this case

$$
-\frac{1}{z-1} = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots .
$$

Thus the Laurent series in region (b) is the sum of these two expressions

$$
\begin{aligned}
f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\cdots .
\end{aligned}
$$

(c) In this region, we are expanding around $z = 1$, so we are seeking a series of the form

$$
f(z) = \sum_{n=-\infty}^{\infty} a_n (z-1)^n .
$$

Since in this region $0 < |z-1| < 1$, so we choose to write the function as

$$
f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{(z-1)\left[(z-1)-1\right]} = -\frac{1}{(z-1)\left[1-(z-1)\right]},
$$

and use the geometric series for

$$\frac{1}{1-(z-1)} = 1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots$$

Therefore the desired Laurent series valid in region (c) is

$$\begin{aligned} f(z) &= -\frac{1}{(z-1)}\left(1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots\right) \\ &= -\frac{1}{(z-1)} - 1 - (z-1) - (z-1)^2 - \cdots . \end{aligned}$$

(d) In this region we are seeking an expansion about $z = 2$ in form of

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-2)^n.$$

that is valid for $|z-2| > 1$. So we choose to write the function as

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{[(z-2)+1](z-2)} = \frac{1}{(z-2)^2\left[1 + \frac{1}{z-2}\right]}.$$

Since $\left|\frac{1}{z-2}\right| < 1$, we can use the geometric series for

$$\left[1 + \frac{1}{z-2}\right]^{-1} = 1 - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots .$$

Therefore the desired Laurent series valid in region (d) is

$$\begin{aligned} f(z) &= \frac{1}{(z-2)^2}\left(1 - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right) \\ &= \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \frac{1}{(z-2)^4} - \frac{1}{(z-2)^5} + \cdots . \end{aligned}$$

3.4 Theory of Residues

3.4.1 Zeros and Poles

Zeros

If $f(z_0) = 0$, then the point z_0 is said to be a zero of the function $f(z)$. If $f(z)$ is analytic at z_0, then we can expand it in a Taylor series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots.$$

Since z_0 is a zero of the function, clearly $a_0=0$. If $a_1\neq 0$, then z_0 is said to be a simple zero. If both a_0 and a_1 are zero and $a_2\neq 0$, then z_0 is a zero of order two, and so on.

If $f(z)$ has a zero of order m at z_0, that is, $a_0,a_1,\ldots,a_{m-1}$ are all zero and $a_m\neq 0$, then $f(z)$ can be written as

$$f(z)=(z-z_0)^m g(z),$$

where

$$g(z)=a_m+a_{m+1}(z-z_0)+a_{m+2}(z-z_0)^2+\cdots.$$

It is clear that $g(z)$ is analytic (therefore continuous) at z_0, and $g(z_0)=a_m\neq 0$. It follows that in the immediate neighborhood of z_0, there is no other zero, because $g(z)$ cannot suddenly drop to zero, since it is continuous. Therefore there exists a disk of finite radius δ surrounds z_0, within which $g(z)\neq 0$. In other words,

$$f(z)\neq 0 \quad \text{for} \quad 0<|z-z_0|<\delta.$$

In this sense, z_0 is said to be an isolated zero of $f(z)$.

Isolated Singularities

As we recall, a singularity of a function $f(z)$ is a point at which $f(z)$ is not analytic. A point at which $f(z)$ is analytic is called a regular point. A point z_0 is said to be an isolated singularity of $f(z)$ if there exists a neighborhood of z_0 in which z_0 is the only singular point of $f(z)$. For example, a rational function $P(z)/Q(z)$, (the ratio of two polynomials), is analytic everywhere except at zeros of $Q(z)$. If all the zeros of $Q(z)$ are isolated, then all the singularities of $P(z)/Q(z)$ are isolated.

Poles

If $f(z)$ has an isolated singular point at z_0, then in the immediate neighborhood of z_0, $f(z)$ can be expanded in a Laurent series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{a_{-1}}{(z-z_0)}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-3}}{(z-z_0)^3}+\cdots.$$

The portion of the series involving negative powers of $(z-z_0)$ is called the principal part of $f(z)$ at z_0. If the principal part contains at least one nonzero term but the number of such terms are finite, then there exists an integer m such that

$$a_{-m} \neq 0 \quad \text{and} \quad a_{-(m+1)} = a_{-(m+2)} = \cdots = 0.$$

That is, the expansion takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-m}}{(z - z_0)^m},$$

where $a_{-m} \neq 0$. In this case, the isolated singular point z_0 is called a pole of order m. A pole of order one is usually referred to as a simple pole.

If an infinite number of coefficients of negative powers are nonzero, then z_0 is called an essential singular point.

3.4.2 Definition of the Residue

If z_0 is an isolated singular point of $f(z)$, then the function $f(z)$ is analytic in the neighborhood of $z = z_0$ with the exception of the point $z = z_0$ itself. In the immediate neighborhood of z_0, $f(z)$ can be expanded in a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{3.19}$$

The coefficients a_n are expressed in terms of contour integrals of (3.13). Among the coefficients, a_{-1} is of particular interest,

$$a_{-1} = \frac{1}{2\pi \mathrm{i}} \oint_C f(z)\, \mathrm{d}z, \tag{3.20}$$

where C is a closed contour in the counterclockwise direction around z_0. It is the coefficient of $(z - z_0)^{-1}$ term in the expansion, and is called the residue of $f(z)$ at the isolated singular point z_0. We emphasize once again, for a_{-1} of (3.20) to be called the residue at z_0, the closed contour C must not contain any singularity other than z_0. We shall denote this residue as

$$a_{-1} = \mathrm{Res}_{z=z_0}\left[f(z)\right].$$

The reason for the name "residue" is that if we integrate the Laurent series term by term over a circular contour, the only term which survives the integration process is the a_{-1} term. This follows from (3.19) that

$$\oint f(z)\, \mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \oint (z - z_0)^n \mathrm{d}z.$$

These integrals can be easily evaluated by setting $z - z_0 = r\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{d}z = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$,

$$\oint (z - z_0)^n \mathrm{d}z = \int_0^{2\pi} \mathrm{i}r^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}\mathrm{d}\theta = \begin{cases} 0 & n \neq -1 \\ 2\pi\mathrm{i} & n = -1. \end{cases}$$

Thus, only the term with $n = -1$ is left. The coefficient of this term is called the residue

$$\frac{1}{2\pi\mathrm{i}} \oint f(z)\,\mathrm{d}z = a_{-1}.$$

3.4.3 Methods of Finding Residues

Residues are defined in (3.20). In some cases, we can carry out this integral directly. However, in general, residues can be found by much easier methods. It is because of these methods, residues are so useful.

Laurent Series

If it is easy to write down the Laurent series for $f(z)$ about $z = z_0$ that is valid in the immediate neighborhood of z_0, then the residue is just the coefficient a_{-1} of the term $1/(z - z_0)$. For example

$$f(z) = \frac{3}{z-2}$$

is already in the form of a Laurent series about $z = 2$ with $a_{-1} = 3$ and $a_n = 0$ for $n \neq -1$. Therefore the residue at 2 is simply 3.

It is also easy to find the residue of $\exp(1/z^2)$ at $z = 0$, since

$$\mathrm{e}^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^6} + \cdots.$$

There is no $1/z$ term, therefore the residue is equal to zero.

Simple Pole

Suppose that $f(z)$ has a simple, or first-order, pole at $z = z_0$, so we can write

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots.$$

If we multiply this identity by $(z - z_0)$, we get

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots.$$

Now if we let z approach z_0, we obtain for the residue

$$a_{-1} = \lim_{z \to a} (z - z_0) f(z).$$

For example, if

$$f(z) = \frac{4 - 3z}{z(z-1)(z-2)},$$

the residue at $z = 0$ is

$$\mathrm{Res}_{z=0}\,[f\,(z)] = \lim_{z \to 0} z \frac{4 - 3z}{z(z-1)(z-2)} = \frac{4}{(-1)\,(-2)} = 2,$$

the residue at $z = 1$ is

$$\mathrm{Res}_{z=1}\,[f\,(z)] = \lim_{z \to 1} (z-1) \frac{4 - 3z}{z(z-1)(z-2)} = \frac{4-3}{1\,(-1)} = -1,$$

and the residue at $z = 2$ is

$$\mathrm{Res}_{z=2}\,[f\,(z)] = \lim_{z \to 2} (z-2) \frac{4 - 3z}{z(z-1)(z-2)} = \frac{4-6}{2\,(1)} = -1.$$

These results can also be understood in terms of partial fractions. It can be readily verified that

$$f(z) = \frac{4 - 3z}{z(z-1)(z-2)} = \frac{2}{z} + \frac{-1}{z-1} + \frac{-1}{z-2}.$$

In the region $|z| < 1$, both $\frac{-1}{z-1}$ and $\frac{-1}{z-2}$ are analytic. Therefore they can be expressed in terms of Taylor series, which has no negative power terms. Thus, the Laurent series of $f(z)$ about $z = 0$ in the region $0 < |z| < 1$ is of the form

$$f(z) = \frac{2}{z} + a_0 + a_1 z + a_2 z^2 + \cdots.$$

It is seen that a_{-1} comes solely from the first term. Therefore the residue at $z = 0$ must equal to 2.

Similarly, the Laurent series of $f(z)$ about $z = 1$ in the region $0 < |z-1| < 1$ is of the form

$$f(z) = \frac{-1}{z-1} + a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots.$$

Hence a_{-1} is equal to -1. For the same reason, the residue at $z = 2$ comes from the term $\frac{-1}{z-2}$, and is clearly equal to -1.

Multiple-Order Pole

If $f(z)$ has a third-order pole at $z = z_0$, then

$$f(z) = \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots .$$

To obtain the residue a_{-1}, we must multiply this identity by $(z-a)^3$

$$(z-z_0)^3 f(z) = a_{-3} + a_{-2}(z-z_0) + a_{-1}(z-z_0)^2 + a_0(z-z_0)^3 + \cdots$$

and differentiate twice with respect to z

$$\frac{\mathrm{d}}{\mathrm{d}z}[(z-z_0)^3 f(z)] = a_{-2} + 2a_{-1}(z-z_0) + 3a_0(z-z_0)^2 + \cdots ,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}[(z-z_0)^3 f(z)] = 2a_{-1} + 3\cdot 2a_0\,(z-z_0) + \cdots .$$

Next we let z approach z_0

$$\lim_{z\to z_0} \frac{\mathrm{d}^2}{\mathrm{d}z^2}[(z-z_0)^3 f(z)] = 2a_{-1},$$

and finally divide it by 2,

$$\frac{1}{2}\lim_{z\to z_0} \frac{\mathrm{d}^2}{\mathrm{d}z^2}[(z-z_0)^3 f(z)] = a_{-1}.$$

Thus, if $f(z)$ has a pole of order m at $z = z_0$, then the residue of $f(z)$ at $z = z_0$ is

$$\mathrm{Res}_{z=z_0}\,[f\,(z)] = \frac{1}{(m-1)!}\lim_{z\to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}[(z-a)^m f(z)].$$

For example,

$$f\,(z) = \frac{1}{z\,(z-2)^4}$$

clearly has a fourth order pole at $z = 2$. Thus

$$\mathrm{Res}_{z=2}\,[f\,(z)] = \frac{1}{3!}\lim_{z\to 2}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\left[(z-2)^4\frac{1}{z\,(z-2)^4}\right]$$

$$= \frac{1}{6}\lim_{z\to 2}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\frac{1}{z} = \frac{1}{6}\lim_{z\to 2}\frac{(-1)\,(-2)\,(-3)}{z^4} = -\frac{1}{16}.$$

To check this result, we can expand $f\,(z)$ in a Laurent series about $z = 2$ in the region $0 < |z-2| < 2$. For this purpose, let us write $f\,(z)$ as

$$f\,(z) = \frac{1}{z\,(z-2)^4} = \frac{1}{(z-2)^4}\frac{1}{[2+(z-2)]} = \frac{1}{2\,(z-2)^4}\frac{1}{\left(1+\frac{z-2}{2}\right)}.$$

Since $\left|\frac{z-2}{2}\right| < 1$, so we have

$$f(z) = \frac{1}{2(z-2)^4}\left[1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\frac{1}{(z-2)^4} - \frac{1}{4}\frac{1}{(z-2)^3} + \frac{1}{8}\frac{1}{(z-2)^2} - \frac{1}{16}\frac{1}{(z-2)} + \frac{1}{32} - \cdots .$$

It is seen that the coefficient of the $(z-2)^{-1}$ is indeed $-1/16$.

Derivative of the Denominator

If $p(z)$ and $q(z)$ are analytic functions, and $q(z)$ has a simple zero at z_0 and $p(z_0) \neq 0$, then

$$f(z) = \frac{p(z)}{q(z)}$$

has a simple pole at z_0. As $q(z)$ is analytic, so it can be expressed as a Taylor series about z_0

$$q(z) = q(z_0) + q'(z_0)(z-z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \cdots .$$

But it has a zero at z_0, so $q(z_0) = 0$, and

$$q(z) = q'(z_0)(z-z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \cdots .$$

Since $f(z)$ has a simple pole at z_0, its residue at z_0 is

$$\begin{aligned}
\operatorname{Res}_{z=z_0}[f(z)] &= \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)} \\
&= \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q'(z_0)(z-z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \cdots} \\
&= \frac{p(z_0)}{q'(z_0)}.
\end{aligned}$$

This formula is very often the most efficient way of finding the residue. For example, the function

$$f(z) = \frac{z}{z^4+4}$$

has four simple poles, located at the zeros of the denominator

$$z^4 + 4 = 0.$$

The four roots of this equation are

$$\begin{aligned}
z_1 &= \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4} = 1+\mathrm{i}, \\
z_2 &= \sqrt{2}\mathrm{e}^{\mathrm{i}(\pi/4+\pi/2)} = -1+\mathrm{i}, \\
z_3 &= \sqrt{2}\mathrm{e}^{\mathrm{i}(\pi/4+\pi)} = -1-\mathrm{i}, \\
z_4 &= \sqrt{2}\mathrm{e}^{\mathrm{i}(\pi/4+3\pi/2)} = 1-\mathrm{i}.
\end{aligned}$$

The residues at $z_1,\ z_2,\ z_3$, and z_4 are

$$\begin{aligned}
\mathrm{Res}_{z=z_1}\left[f(z)\right] &= \lim_{z\to z_1}\frac{z}{\left(z^4+4\right)'} = \lim_{z\to z_1}\frac{z}{4z^3} = \lim_{z\to z_1}\frac{1}{4z^2} \\
&= \lim_{z\to(1+\mathrm{i})}\frac{1}{4z^2} = \frac{1}{4\left(1+\mathrm{i}\right)^2} = -\frac{1}{8}\mathrm{i}, \\
\mathrm{Res}_{z=z_2}\left[f(z)\right] &= \lim_{z\to(-1+\mathrm{i})}\frac{1}{4z^2} = \frac{1}{4\left(-1+\mathrm{i}\right)^2} = \frac{1}{8}\mathrm{i}, \\
\mathrm{Res}_{z=z_3}\left[f(z)\right] &= \lim_{z\to(-1-\mathrm{i})}\frac{1}{4z^2} = \frac{1}{4\left(-1-\mathrm{i}\right)^2} = -\frac{1}{8}\mathrm{i}, \\
\mathrm{Res}_{z=z_4}\left[f(z)\right] &= \lim_{z\to(1-\mathrm{i})}\frac{1}{4z^2} = \frac{1}{4\left(1-\mathrm{i}\right)^2} = \frac{1}{8}\mathrm{i}.
\end{aligned}$$

It can be readily verified that

$$\frac{z}{z^4+4} = \frac{-\mathrm{i}/8}{z-(1+\mathrm{i})} + \frac{\mathrm{i}/8}{z-(-1+\mathrm{i})} + \frac{-\mathrm{i}/8}{z-(-1-\mathrm{i})} + \frac{\mathrm{i}/8}{z-(1-\mathrm{i})}.$$

Therefore the calculated residues must be correct.

3.4.4 Cauchy's Residue Theorem

Consider a simple closed curve C containing in its interior a number of isolated singular points, $z_1,\ z_2,\ \ldots,$ of a function $f(z)$. If around each singular point we draw a circle so small that it encloses no other singular points as shown in Fig. 3.6, so $f(z)$ is analytic in the region between C and these small circles. Then introducing cuts as in the proof of Laurent series, we find by the Cauchy theorem that the integral around C counterclockwise plus the integral around the small circles clockwise is zero, since the integrals along the cuts cancel. Thus

$$\frac{1}{2\pi\mathrm{i}}\oint_C f(z)\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\oint_{C_1} f(z)\mathrm{d}z + \cdots - \frac{1}{2\pi\mathrm{i}}\oint_{C_n} f(z)\mathrm{d}z = 0,$$

where all integrals are counterclockwise, the minus sign is to account for the clockwise direction of the small circles. It follows that:

$$\frac{1}{2\pi\mathrm{i}}\oint_C f(z)\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\oint_{C_1} f(z)\mathrm{d}z + \cdots + \frac{1}{2\pi\mathrm{i}}\oint_{C_n} f(z)\mathrm{d}z.$$

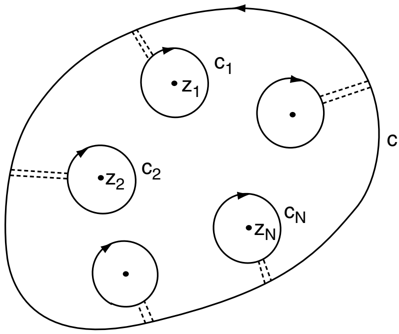


Fig. 3.6. The circles $C_1, C_2, \ldots, C_N$ enclosing, respectively, the singular points $z_1, z_2, \ldots, z_N$ within a simple closed curve

The integrals on the right are, by definition, just the residues of $f(z)$ at the various isolated singularities within C. Hence we have established the important residue theorem:

If there are n number of singular points of $f(z)$ inside the contour C, then

$$\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\left\{\mathrm{Res}_{z=z_1}[f(z)] + \mathrm{Res}_{z=z_2}[f(z)] + \cdots + \mathrm{Res}_{z=z_n}[f(z)]\right\}. \tag{3.21}$$

This theorem is known as Cauchy's residue theorem or just the residue theorem.

3.4.5 Second Residue Theorem

If the number of singular points inside the enclosed contour C is too large, or there are nonisolated singular points interior in C, it will be difficult to carry out the contour integration with the Cauchy's residue theorem. For such cases, there is another residue theorem that is more efficient.

Suppose $f(z)$ has many singular points in C and no singular point outside of C, as shown in Fig. 3.7.

If we want to evaluate the integral $\oint_C f(z)\,\mathrm{d}z$, we can first construct a circular contour C_R outside of C, centered at the origin with a radius R. Then by the principle of deformation of contours

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_R} f(z)\,\mathrm{d}z.$$

Now if we expand $f(z)$ in terms of Laurent series about $z = 0$ in the region $|z| > R$,

$$f(z) = \cdots + \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots,$$

the coefficient a_{-1} is given by the integral

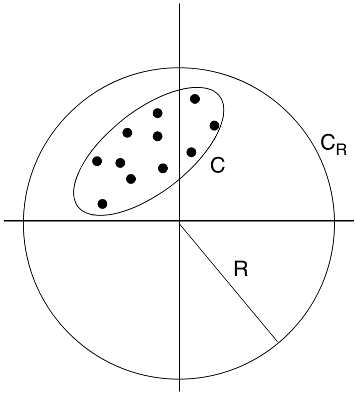


Fig. 3.7. If the number of singularities enclosed in C is too large, then it is more convenient to replace the contour C with a large circular contour C_R centered at the origin

$$a_{-1} = \frac{1}{2\pi \mathrm{i}} \oint_{C_R} f(z)\,\mathrm{d}z.$$

Note that a_{-1} in this equation is not the residue of $f(z)$ about $z = 0$, because the series is not valid in the immediate neigborhood of $z = 0$. However, if we change z to $1/z$, then

$$f\left(\frac{1}{z}\right) = \cdots + a_{-3}z^3 + a_{-2}z^2 + a_{-1}z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots,$$

is convergent for $|z| < 1/R$. It is seen that a_{-1} is the residue at $z = 0$ of the function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$, since

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \cdots + a_{-3}z + a_{-2} + \frac{a_{-1}}{z} + \frac{a_0}{z^2} + \frac{a_1}{z^3} + \cdots,$$

is a Laurent series valid in the region $0 < |z| < \frac{1}{R}$. Hence we arrived at the following theorem.

If $f(z)$ is analytic everywhere except for a number of singular points interior to a positive oriented closed contour C, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi \mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

Example 3.4.1. Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ for

$$f(z) = \frac{5z-2}{z(z-1)},$$

where C is along the circle $|z| = 2$ in the counterclockwise direction. (a) Use the Cauchy residue theorem. (b) Use the second residue theorem.

Solution 3.4.1. (a) The function has two single poles at $z = 0,\ z = 1$. Both lie within the circle $|z| = 2$. So

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left\{\mathrm{Res}_{z=0}[f(z)] + \mathrm{Res}_{z=1}[f(z)]\right\}.$$

Since

$$\mathrm{Res}_{z=0}[f(z)] = \lim_{z\to 0} z\frac{5z-2}{z(z-1)} = \frac{-2}{-1} = 2,$$
$$\mathrm{Res}_{z=1}[f(z)] = \lim_{z\to 1}(z-1)\frac{5z-2}{z(z-1)} = \frac{3}{1} = 3,$$

thus

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,(2+3) = 10\pi\mathrm{i}.$$

If C is in the clockwise direction, the answer would be $-10\pi\mathrm{i}$.

(b) According to the second residue theorem

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right].$$

Now

$$f\left(\frac{1}{z}\right) = \frac{5/z-2}{1/z(1/z-1)} = \frac{(5-2z)z}{1-z},$$

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{5-2z}{z(1-z)},$$

which has a simple pole at $z = 0$. Thus

$$\mathrm{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right] = \lim_{z\to 0} z\frac{5-2z}{z(1-z)} = \frac{5}{1} = 5.$$

Therefore

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\cdot 5 = 10\pi\mathrm{i}.$$

Not surprisingly, this is the same result obtained in (a).

Example 3.4.2. Find the value of the integral

$$\oint_C \frac{\mathrm{d}z}{z^3(z+4)}$$

taken counterclockwise around the circle (a) $|z| = 2$, (b) $|z+2| = 3$.

Solution 3.4.2. (a) The function has a third-order pole at $z = 0$ and a simple pole at $z = -4$. Only $z = 0$ is inside the circle $|z| = 2$. Therefore

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i}\, \mathrm{Res}_{z=0}[f(z)].$$

For the third-order pole,

$$\mathrm{Res}_{z=0}[f(z)] = \frac{1}{2!} \lim_{z \to 0} \frac{\mathrm{d}^2}{\mathrm{d}z^2} z^3 \frac{1}{z^3 (z+4)} = \frac{1}{2} \lim_{z \to 0} \frac{2}{(z+4)^3} = \frac{1}{64}.$$

Therefore

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i} \frac{1}{64} = \frac{\pi}{32} \mathrm{i}.$$

(b) For the circle $|z+2| = 3$, the center is at $z = -2$ and the radius is 3. Both singular points are inside the circle. Thus

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i} \left\{ \mathrm{Res}_{z=0}[f(z)] + \mathrm{Res}_{z=-4}[f(z)] \right\}.$$

Since

$$\mathrm{Res}_{z=-4}[f(z)] = \lim_{z \to -4} (z+4) \frac{1}{z^3 (z+4)} = \frac{1}{(-4)^3} = -\frac{1}{64},$$

so

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i} \left\{ \frac{1}{64} - \frac{1}{64} \right\} = 0.$$

Example 3.4.3. Find the value of the integral

$$\oint_C \tan \pi z \mathrm{d}z$$

taken counterclockwise around the unit circle $|z| = 1$.

Solution 3.4.3. Since

$$f(z) = \tan \pi z = \frac{\sin \pi z}{\cos \pi z},$$

and

$$\cos \frac{2n+1}{2} \pi = 0, \quad \text{for} \quad n = \ldots, -2, -1, 0, 1, 2, \ldots,$$

therefore $z = (2n+1)/2$ are zeros of $\cos \pi z$. Expanding $\cos \pi z$ about any of these zeros in Taylor series, one can readily see that $f(z)$ has a simple pole at each of these singular points. Among them, $z = 1/2$ and $z = -\,1/2$ are inside $|z| = 1$. Hence

$$\oint_C \tan \pi z \mathrm{d}z = 2\pi\mathrm{i}\left\{\mathrm{Res}_{z=1/2}[f(z)] + \mathrm{Res}_{z=-1/2}[f(z)]\right\}.$$

The simplest way to find these residues is by the "derivative of the denominator" method,

$$\mathrm{Res}_{z=1/2}[f(z)] = \left[\frac{\sin \pi z}{(\cos \pi z)'}\right]_{z=\frac{1}{2}} = \left[\frac{\sin \pi z}{-\pi \sin \pi z}\right]_{z=\frac{1}{2}} = -\frac{1}{\pi},$$

$$\mathrm{Res}_{z=-1/2}[f(z)] = \left[\frac{\sin \pi z}{(\cos \pi z)'}\right]_{z=-\frac{1}{2}} = \left[\frac{\sin \pi z}{-\pi \sin \pi z}\right]_{z=-\frac{1}{2}} = -\frac{1}{\pi}.$$

Therefore

$$\oint_C \tan \pi z \mathrm{d}z = 2\pi\mathrm{i}\left\{-\frac{1}{\pi} - \frac{1}{\pi}\right\} = -4\mathrm{i}.$$

Example 3.4.4. Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ for

$$f(z) = z^2 \exp\left(\frac{1}{z}\right),$$

where C is counterclockwise around the unit circle $|z| = 1$.

Solution 3.4.4. The function $f(z)$ has an essential singularity at $z = 0$. Thus

$$\oint_C z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}[f(z)].$$

The residue is simply the coefficient of the z^{-1} term in the Laurent series about $z = 0$,

$$\begin{aligned} z^2 \exp\left(\frac{1}{z}\right) &= z^2\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots\right) \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots . \end{aligned}$$

Therefore

$$\mathrm{Res}_{z=0}[f(z)] = \frac{1}{3!} = \frac{1}{6}.$$

Hence

$$\oint_C z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi\mathrm{i}\frac{1}{6} = \frac{\pi}{3}\mathrm{i}.$$

Example 3.4.5. Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ for

$$f(z) = \frac{z^{99}\exp\left(\frac{1}{z}\right)}{z^{100}+1},$$

where C is counterclockwise around the circle $|z| = 2$.

Solution 3.4.5. There are 100 singular points located on the circumference of the unit circle $|z| = 1$ and an essential singular point at $z = 0$. Obviously the second residue theorem is more convenient. That is,

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

Now

$$f\left(\frac{1}{z}\right) = \frac{(1/z)^{99}\exp(z)}{(1/z)^{100}+1} = \frac{z\exp(z)}{1+z^{100}},$$
$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\exp(z)}{z(1+z^{100})}.$$

So

$$\mathrm{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = \lim_{z\to 0} z\frac{\exp(z)}{z(1+z^{100})} = 1.$$

Therefore

$$\oint_C \frac{z^{99}\exp\left(\frac{1}{z}\right)}{z^{100}+1}\,\mathrm{d}z = 2\pi\mathrm{i}.$$

Example 3.4.6. (a) Show that if $z = 1$ and $z = 2$ are inside the closed contour C, then

$$\oint_C \frac{1}{(z-1)(z-2)}\,\mathrm{d}z = 0.$$

(b) Show that if all the singular points $s_1, s_2, \ldots, s_n$ of the following function

$$f(z) = \frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)}$$

are inside the closed contour C, then

$$I = \oint_C f(z)\,\mathrm{d}z = 0.$$

Solution 3.4.6. (a) Taking partial fraction, we have

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}.$$

So

$$\oint_C \frac{1}{(z-1)(z-2)}\,\mathrm{d}z = \oint_C \frac{A}{(z-1)}\,\mathrm{d}z + \oint_C \frac{B}{(z-2)}\,\mathrm{d}z$$
$$= 2\pi\mathrm{i}\,(A+B)\,.$$

Since

$$\frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)} = \frac{(A+B)z-(2A+B)}{(z-1)(z-2)}$$

and

$$\frac{(A+B)z-(2A+B)}{(z-1)(z-2)} = \frac{1}{(z-1)(z-2)},$$

it follows that:

$$A+B=0.$$

Therefore

$$\oint_C \frac{1}{(z-1)(z-2)}\,\mathrm{d}z = 0.$$

(b) The partial fraction of $f(z)$ is of the form

$$\frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)} = \frac{r_1}{(z-s_1)} + \frac{r_2}{(z-s_2)} + \cdots + \frac{r_n}{(z-s_n)}.$$

Therefore

$$\oint_C f(z)\,\mathrm{d}z = \oint_C \frac{r_1}{(z-s_1)}\,\mathrm{d}z + \oint_C \frac{r_2}{(z-s_2)}\,\mathrm{d}z + \cdots + \oint_C \frac{r_n}{(z-s_n)}\,\mathrm{d}z$$
$$= 2\pi\mathrm{i}\,(r_1+r_2+\cdots+r_n)\,.$$

Now

$$\frac{r_1}{(z-s_1)} + \frac{r_2}{(z-s_2)} + \cdots + \frac{r_n}{(z-s_n)} = \frac{(r_1+r_2+\cdots+r_n)\,z^{n-1}+\cdots}{(z-s_1)(z-s_2)\cdots(z-s_n)},$$

and

$$\frac{(r_1+r_2+\cdots+r_n)\,z^{n-1}+\cdots}{(z-s_1)(z-s_2)\cdots(z-s_n)} = \frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)}.$$

Since the numerator of the right-hand side has no z^{n-1} term, therefore

$$(r_1+r_2+\cdots+r_n)=0,$$

and

$$\oint_C \frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)}\mathrm{d}z = 0.$$

3.5 Evaluation of Real Integrals with Residues

A very surprising fact is that we can use the residue theorem to evaluate integrals of real variable. For certain classes of complicated real integrals, residue theorem offers a simple and elegant way of carrying out the integration.

3.5.1 Integrals of Trigonometric Functions

Let us consider the integral of the type

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\mathrm{d}\theta.$$

If we make the substitution

$$z = \mathrm{e}^{\mathrm{i}\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}z,$$

then

$$\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

and

$$\mathrm{d}\theta = \frac{1}{\mathrm{i}z}\,\mathrm{d}z.$$

The given integral takes the form

$$I = \oint_C f(z)\frac{\mathrm{d}z}{\mathrm{i}z},$$

the integration being taken counterclockwise around the unit circle centered at $z = 0$.

We illustrate this method with following examples.

Example 3.5.1. Show that

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{2} - \cos\theta} = 2\pi.$$

Solution 3.5.1. With the transformation just discussed we can write the integral as

$$I = \oint_C \frac{\mathrm{d}z}{\left[\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)\right] \mathrm{i}z} = \oint_C \frac{-2\,\mathrm{d}z}{\mathrm{i}(z^2 - 2\sqrt{2}z + 1)}$$
$$= -\frac{2}{\mathrm{i}} \oint_C \frac{\mathrm{d}z}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

The integrand has two simple poles. The one at $\sqrt{2} + 1$ lies outside the unit circle and is thus of no interest. The one at $\sqrt{2} - 1$ is inside the unit circle, and the residue at that point is

$$\mathrm{Res}_{z=\sqrt{2}-1}\left[f(z)\right] = \lim_{z \to \sqrt{2}-1} (z - \sqrt{2} + 1) \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = -\frac{1}{2}.$$

Thus

$$I = -\frac{2}{\mathrm{i}} 2\pi \mathrm{i} \left(-\frac{1}{2}\right) = 2\pi.$$

Example 3.5.2. Evaluate the integral

$$I = \int_0^{\pi} \frac{\mathrm{d}\theta}{a - b\cos\theta}, \qquad a > b > 0.$$

Solution 3.5.2. Since the integrand is symmetric about $\theta = \pi$, so we can extend the integration interval to $[0, 2\pi]$,

$$I = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}\theta}{a - b\cos\theta},$$

which can be written as an integral around an unit circle in the complex plane

$$I = \frac{1}{2} \oint \frac{1}{a - b\frac{1}{2}(z + \frac{1}{z})} \frac{\mathrm{d}z}{\mathrm{i}z} = \oint \frac{1}{2az - bz^2 - b} \frac{\mathrm{d}z}{\mathrm{i}}.$$

Now

$$\oint \frac{1}{2az - bz^2 - b} \frac{\mathrm{d}z}{\mathrm{i}} = -\frac{1}{b\mathrm{i}} \oint \frac{1}{z^2 - \frac{2a}{b}z + 1}\,\mathrm{d}z,$$

taking this seemingly trivial step of making the coefficient of z^2 to be 1 can actually avoid many pitfalls of what follows. The singular points of the integrand are at the zeros of the denominator,

$$z^2 - \frac{2a}{b}z + 1 = 0.$$

Let z_1 and z_2 be the roots of this equation. They are easily found to be

$$z_1 = \frac{1}{b}\left(a - \sqrt{a^2 - b^2}\right), \qquad z_2 = \frac{1}{b}\left(a + \sqrt{a^2 - b^2}\right).$$

Since

$$(z - z_1)(z - z_2) = z^2 - (z_1 + z_2) + z_1 z_2 = z^2 - \frac{2a}{b}z + 1,$$

it follows that:

$$z_1 z_2 = 1.$$

This means that one root must be greater than 1, and the other less than 1. Furthermore, $z_1 < z_2$, z_1 must be less than 1 and z_2 greater than 1. Therefore only z_1 is inside the unit circle. Thus

$$\begin{aligned}\oint \frac{\mathrm{d}z}{(z - z_1)(z - z_2)} &= 2\pi\mathrm{i}\,\mathrm{Res}_{z=z_1}[f(z)] \\ &= 2\pi\mathrm{i} \lim_{z \to z_1} (z - z_1) \frac{1}{(z - z_1)(z - z_2)} = 2\pi\mathrm{i}\frac{1}{(z_1 - z_2)},\end{aligned}$$

and

$$\frac{1}{(z_1 - z_2)} = -\frac{b}{2\sqrt{a^2 - b^2}}.$$

Therefore

$$I = -\frac{1}{b\mathrm{i}}2\pi\mathrm{i}\left(-\frac{b}{2\sqrt{a^2 - b^2}}\right) = \frac{\pi}{\sqrt{a^2 - b^2}}.$$

Example 3.5.3. Show that

$$\int_0^{2\pi} \cos^{2n}\theta \mathrm{d}\theta = \frac{2\pi\,(2n)!}{2^{2n}\,(n!)^2}.$$

Solution 3.5.3. The integral can be written as

$$I = \int_0^{2\pi} \cos^{2n}\theta \mathrm{d}\theta = \oint_C \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^{2n} \frac{\mathrm{d}z}{\mathrm{i}z} = \frac{1}{2^{2n}\mathrm{i}} \oint_C \left[\sum_{k=0}^{2n} C_k^{2n} z^{2n-k} \frac{1}{z^k}\right] \frac{\mathrm{d}z}{z},$$

where C_k^{2n} are the binomial coefficients

$$C_k^{2n} = \frac{(2n)!}{k!(2n-k)!}.$$

Carrying out the integration term by term, the only nonvanishing term is the term of z^{-1}. Since

$$\left[\sum_{k=0}^{2n} C_k^{2n} z^{2n-k} \frac{1}{z^k}\right] \frac{1}{z} = \left[\sum_{k=0}^{2n} C_k^{2n} z^{2n-2k}\right] \frac{1}{z},$$

it is clear that the coefficient of z^{-1} is given by term with $k = n$. Thus

$$\begin{aligned} I &= \frac{1}{2^{2n}\mathrm{i}} \oint_C \left[z^{2n-1} + 2nz^{2n-3} + \cdots + \frac{C_n^{2n}}{z} + \cdots + \frac{1}{z^{2n+1}}\right] \mathrm{d}z \\ &= \frac{1}{2^{2n}\mathrm{i}} 2\pi\mathrm{i}\, C_n^{2n} = \frac{2\pi\,(2n)!}{2^{2n}(n!)^2}. \end{aligned}$$

3.5.2 Improper Integrals I: Closing the Contour with a Semicircle at Infinity

We consider the real integrals of the type

$$I = \int_{-\infty}^{\infty} f(x)\mathrm{d}x.$$

Such an integral, for which the interval of integration is not finite, is called improper integral, and it has the meaning

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}x.$$

Under certain conditions, this type of integral can be evaluated with the residue theorem. The idea is to close the contour by adding additional pieces along which the integral is either zero or some multiple of the original integral along the real axis.

If $f(x)$ is a rational function (i.e., ratio of two polynomials) with no singularity on the real axis and

$$\lim_{z\to\infty} zf(z) = 0,$$

then it can be shown that the integral along the real axis from $-\infty$ to ∞ is equal to the integral around a closed contour which consists of (a) the straight line along the real axis and (b) the semicircle C_R at infinity as shown in Fig. 3.8.

This is so, because with

$$z = R\mathrm{e}^{\mathrm{i}\theta}, \qquad \mathrm{d}z = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}z\mathrm{d}\theta,$$

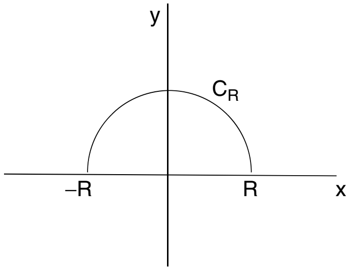


Fig. 3.8. As $R \to \infty$, the semicircle C_R is at infinity. The contour consists of the real axis and C_R encloses the entire upper half-plane

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| = \left|\int_0^{\pi} f(z)\,\mathrm{i}z\mathrm{d}\theta\right| \leq \mathrm{Max}\,|f(z)\,z|\,\pi,$$

which goes to zero as $R \to \infty$, since $\lim_{z\to\infty} zf(z) = 0$. Therefore

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

It follows that:

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty}\left[\int_{-R}^{R} f(x)\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z\right] = \oint_{\mathrm{u.h.p}} f(z)\,\mathrm{d}z,$$

where u.h.p means the entire upper half-plane. As $R \to \infty$, all the poles of $f(z)$ in the upper half-plane will be inside the contour. Hence

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi\mathrm{i}\,(\text{sum of residues of } f(z) \text{ in the upper half-plane}).$$

By the same token, we can, of course, close the contour in the lower half-plane. However, in that case, the direction of integration will be clockwise. Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \oint_{\mathrm{l.h.p}} f(z)\,\mathrm{d}z \\ &= -2\pi\mathrm{i}\,(\text{sum of residues of } f(z) \text{ in the lower half-plane}).\end{aligned}$$

Example 3.5.4. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$$

Solution 3.5.4. Since

$$\lim_{z\to\infty} z\frac{1}{1+z^2} = 0,$$

we can evaluate this integral with contour integration. That is

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \oint_{\mathrm{u.h.p}} \frac{1}{1+z^2}\,\mathrm{d}z$$

The singular points of

$$f(z) = \frac{1}{1+z^2}$$

are at $z = \mathrm{i}$ and $z = -\mathrm{i}$. Only $z = \mathrm{i}$ is in the upper half-plan. Therefore

$$I = 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}}[f(z)] = 2\pi\mathrm{i}\lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \pi.$$

Now, if we close the contour in the lower half-plane, then

$$I = -2\pi\mathrm{i}\,\mathrm{Res}_{z=-\mathrm{i}}[f(z)] = -2\pi\mathrm{i}\lim_{z\to-\mathrm{i}}(z+\mathrm{i})\frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \pi,$$

which is, of course, the same result.

Example 3.5.5. Show that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

Solution 3.5.5. The four singular points of the function

$$f(z) = \frac{1}{z^4+1}$$

are $\mathrm{e}^{\mathrm{i}\pi/4}$, $\mathrm{e}^{\mathrm{i}3\pi/4}$, $\mathrm{e}^{\mathrm{i}5\pi/4}$, $\mathrm{e}^{\mathrm{i}7\pi/4}$. Only two, $\mathrm{e}^{\mathrm{i}\pi/4}$, $\mathrm{e}^{\mathrm{i}3\pi/4}$ are in the upper half-plane. Therefore

$$\oint_{\mathrm{u.h.p.}} \frac{1}{z^4+1}\,\mathrm{d}z = 2\pi\mathrm{i}\{\mathrm{Res}_{z=\exp(\mathrm{i}\pi/4)}[f(z)] + \mathrm{Res}_{z=\exp(\mathrm{i}3\pi/4)}[f(z)]\}.$$

For problems of this type, it is much easier to find the residue by the method of $p(a)/q'(a)$. If we use the method of $\lim_{z\to a}(z-a)f(z)$, the calculation will be much more cumbersome. Since

$$\mathrm{Res}_{z=\exp(\mathrm{i}\pi/4)}[f(z)] = \left[\frac{1}{(z^4+1)'}\right]_{z=\mathrm{e}^{\mathrm{i}\pi/4}} = \left[\frac{1}{4z^3}\right]_{z=\exp(\mathrm{i}\pi/4)} = \frac{1}{4}\mathrm{e}^{-\mathrm{i}3\pi/4},$$

$$\mathrm{Res}_{z=\exp(\mathrm{i}3\pi/4)}[f(z)] = \left[\frac{1}{4z^3}\right]_{z=\exp(\mathrm{i}3\pi/4)} = \frac{1}{4}\mathrm{e}^{-\mathrm{i}9\pi/4} = \frac{1}{4}\mathrm{e}^{-\mathrm{i}\pi/4},$$

so

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = 2\pi\mathrm{i}\left[\frac{1}{4}\mathrm{e}^{-\mathrm{i}3\pi/4} + \frac{1}{4}\mathrm{e}^{-\mathrm{i}\pi/4}\right]$$
$$= \frac{\pi}{2}\left[\mathrm{e}^{-\mathrm{i}\pi/4} + \mathrm{e}^{\mathrm{i}\pi/4}\right] = \pi\cos\left(\frac{\pi}{4}\right)$$

3.5.3 Fourier Integral and Jordan's Lemma

Another very important class of integrals of the form

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\mathrm{d}x$$

can also be evaluated with the residue theorem. This class is known as the Fourier integral of $f(x)$. We will show that as long as $f(x)$ has no singularity along the real axis and

$$\lim_{z\to\infty} f(z) = 0, \tag{3.22}$$

the contour of this integral can be closed with an infinitely large semicircle in the upper half-plane if k is positive, and in the lower half-plane if k is negative. This statement is based on the Jordan's lemma, which states that, under the condition (3.22),

$$\lim_{R\to\infty} \int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z = 0,$$

where k is a positive real number and C_R is the semicircle in the upper half-plane with infinitely large radius R.

To prove this lemma, we first make the following observation. In Fig. 3.9, $y = \sin\theta$ and $y = \frac{2}{\pi}\theta$ are shown together. It is seen that in the interval $[0, \frac{\pi}{2}]$, the curve $y = \sin\theta$ is concave and always lies on or above the straight line $y = \frac{2}{\pi}\theta$. Therefore

$$\sin\theta \geq \frac{2}{\pi}\theta \quad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

With

$$z = R\mathrm{e}^{\mathrm{i}\theta} = R\cos\theta + \mathrm{i}R\sin\theta, \qquad \mathrm{d}z = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta,$$

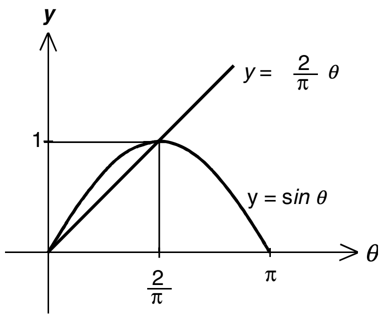


Fig. 3.9. Visualization of the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$

we have

$$\left|\int_{C_R} \mathrm{e}^{ikz} f(z)\,\mathrm{d}z\right| = \left|\int_0^{\pi} \mathrm{e}^{ikz} f(z)\,\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\right| \leq \int_0^{\pi} \left|\mathrm{e}^{ikz}\right| |f(z)|\, R\left|\mathrm{e}^{\mathrm{i}\theta}\right| \mathrm{d}\theta.$$

Since

$$\left|\mathrm{e}^{ikz}\right| = \left|\mathrm{e}^{\mathrm{i}k(R\cos\theta+\mathrm{i}R\sin\theta)}\right| = \left|\mathrm{e}^{\mathrm{i}kR\cos\theta}\right| \left|\mathrm{e}^{-kR\sin\theta}\right| = \mathrm{e}^{-kR\sin\theta},$$

so

$$\left|\int_{C_R} \mathrm{e}^{ikz} f(z)\,\mathrm{d}z\right| \leq \mathrm{Max}\,|f(z)|\, R\int_0^{\pi} \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta.$$

Using $\sin(\pi-\theta) = \sin\theta$, we can write the last integral as

$$\int_0^{\pi} \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta = 2\int_0^{\pi/2} \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta.$$

Now, $\sin\theta \geq \frac{2}{\pi}\theta$ in the interval $[0,\pi/2]$, therefore

$$2\int_0^{\pi/2} \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta \leq 2\int_0^{\pi/2} \mathrm{e}^{-kR2\theta/\pi}\mathrm{d}\theta = \frac{\pi}{kR}\left(1-\mathrm{e}^{-kR}\right). \tag{3.23}$$

Thus

$$\left|\int_{C_R} \mathrm{e}^{ikz} f(z)\,\mathrm{d}z\right| \leq \mathrm{Max}\,|f(z)|\,\frac{\pi}{k}\left(1-\mathrm{e}^{-kR}\right).$$

As $z\to\infty$, $R\to\infty$ and the right-hand side of the last equation goes to zero, since $\lim_{z\to\infty} f(z) = 0$. It follows that:

$$\lim_{z\to\infty}\int_{C_R} \mathrm{e}^{ikz} f(z)\,\mathrm{d}z = 0,$$

and Jordan's lemma is proved. By virtue of this lemma, the Fourier integral can be written as

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\left(\int_{-R}^{R} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x + \int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z\right)$$
$$= \oint_{\mathrm{u.h.p}} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{i=1}^{\mathrm{all}} R_{\mathrm{u.h.p}}\left[\mathrm{e}^{\mathrm{i}kz} f(z)\right],$$

where $\sum_{i=1}^{\mathrm{all}} R_{\mathrm{u.h.p}}\left[\mathrm{e}^{\mathrm{i}kz} f(z)\right]$ means the sum of all residues of $\mathrm{e}^{\mathrm{i}kz} f(z)$ in the upper half-plane.

Note that if k is negative, we cannot close the contour in the upper half-plane, since in (3.23) the factor e^{-kR} will blow up. However, in this case we can close the contour in the lower half-plane, because integrating from $\theta = 0$ to $\theta = -\pi$ will introduce another minus sign to make the large semicircular integral in the lower half-plane vanish. Therefore

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}|k|x} f(x)\,\mathrm{d}x = -2\pi\mathrm{i}\sum_{i=1}^{\mathrm{all}} R_{\mathrm{l.h.p}}\left[\mathrm{e}^{-\mathrm{i}|k|z} f(z)\right],$$

where $\sum_{i=1}^{\mathrm{all}} R_{\mathrm{l.h.p}}\left[\mathrm{e}^{-\mathrm{i}|k|z} f(z)\right]$ means the sum of all residues of $\mathrm{e}^{-\mathrm{i}|k|z} f(z)$ in the lower half-plane. The minus sign is due to the fact that in this case the closed contour integration is clockwise.

Since $\sin kx$ and $\cos kx$ are linear combinations of $\mathrm{e}^{\mathrm{i}kx}$ and $\mathrm{e}^{-\mathrm{i}kx}$, the real integrals of the form

$$\int_{-\infty}^{\infty} \cos kx f(x)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \sin kx f(x)\,\mathrm{d}x$$

can be obtained easily from this class of integrals,

$$\int_{-\infty}^{\infty} \cos kx f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x + \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x\right], \quad (3.24)$$
$$\int_{-\infty}^{\infty} \sin kx f(x)\,\mathrm{d}x = \frac{1}{2\mathrm{i}}\left[\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x\right]. \quad (3.25)$$

If it is certain that the result of the integration is a finite real value, then we may write

$$\int_{-\infty}^{\infty} \cos kx f(x)\,\mathrm{d}x = \mathrm{Re}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x, \quad (3.26)$$
$$\int_{-\infty}^{\infty} \sin kx f(x)\,\mathrm{d}x = \mathrm{Im}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\mathrm{d}x. \quad (3.27)$$

These formulae must be used with caution. While (3.24) and (3.25) are always valid, (3.26) and (3.27) are valid only if there is no imaginary number in $f(x)$.

Example 3.5.6. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x + \mathrm{i}} \,\mathrm{d}x.$$

Solution 3.5.6. There is a simple pole located in the lower half-plane, and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x + \mathrm{i}} \,\mathrm{d}x = \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x + \mathrm{i}} \,\mathrm{d}x - \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}x}}{x + \mathrm{i}} \,\mathrm{d}x.$$

To evaluate the first integral in the right-hand side, we must close the contour in the upper half-plane as shown in Fig. 3.10a.

Since the function is analytic everywhere in the upper half-plane, therefore

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x + \mathrm{i}} \,\mathrm{d}x = \oint_{\mathrm{u.h.p}} \frac{\mathrm{e}^{\mathrm{i}z}}{z + \mathrm{i}} \,\mathrm{d}z = 0.$$

To evaluate the second integral, we must close the contour in the lower half-plane as shown in Fig. 3.10b. Since there is a simple pole located at $z = -\mathrm{i}$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}x}}{x + \mathrm{i}} \,\mathrm{d}x &= \oint_{\mathrm{l.h.p}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z + \mathrm{i}} \,\mathrm{d}z = -2\pi\mathrm{i}\,\mathrm{Res}_{z=-\mathrm{i}} \left[\frac{\mathrm{e}^{-\mathrm{i}z}}{z + \mathrm{i}} \right] \\ &= -2\pi\mathrm{i} \lim_{z \to -\mathrm{i}} (z + \mathrm{i}) \frac{\mathrm{e}^{-\mathrm{i}z}}{z + \mathrm{i}} = -2\pi\mathrm{i}\mathrm{e}^{-1}. \end{aligned}$$

Thus

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x + \mathrm{i}} \,\mathrm{d}x = \frac{1}{2\mathrm{i}} \left[0 - \left(-2\pi\mathrm{i}\mathrm{e}^{-1} \right) \right] = \frac{\pi}{e}.$$

Clearly

$$I \neq \mathrm{Im} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x + \mathrm{i}} \,\mathrm{d}x,$$

this is because there is the imaginary number i in the function.

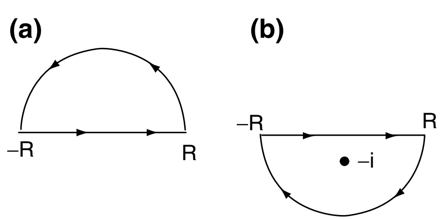


Fig. 3.10. Closing the contour with an infinitely large semicircle. (**a**) contour closed in the upper half-plane, (**b**) contour closed in the lower half-plane

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Example 3.5.7. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2 + 4} \, \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{d}x, \qquad \omega > 0.$$

Solution 3.5.7.

$$I = \oint_{\text{l.h.p}} \frac{1}{z^2 + 4} \, \mathrm{e}^{-\mathrm{i}\omega z} \mathrm{d}z.$$

The only singular point in the lower half-plane is at $z = -2\mathrm{i}$, therefore

$$\begin{aligned} I &= -2\pi\mathrm{i} \operatorname{Res}_{z=-2\mathrm{i}} \left[\frac{\mathrm{e}^{-\mathrm{i}\omega z}}{z^2 + 4} \right] \\ &= -2\pi\mathrm{i} \lim_{z \to -2\mathrm{i}} (z + 2\mathrm{i}) \frac{\mathrm{e}^{-\mathrm{i}\omega z}}{(z + 2\mathrm{i})(z - 2\mathrm{i})} = -2\pi\mathrm{i} \frac{\mathrm{e}^{-2\omega}}{-4\mathrm{i}} = \frac{\pi}{2} \mathrm{e}^{-2\omega}. \end{aligned}$$

This integral happens to be the Fourier transform of $\frac{1}{x^2+4}$, as we shall see in a later chapter.

Example 3.5.8. Evaluate the integral

$$I(t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{R + \mathrm{i}\omega L} \, \mathrm{d}\omega,$$

for both $t > 0$ and $t < 0$.

Solution 3.5.8.

$$I = \frac{A}{2\pi} \frac{1}{\mathrm{i}L} \omega L \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{\left(\frac{R}{\mathrm{i}L}\right) + \omega} \, \mathrm{d}\omega = \frac{A}{2\pi} \frac{1}{\mathrm{i}L} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega - \mathrm{i}\frac{R}{L}} \, \mathrm{d}\omega.$$

For $t > 0$, we can close the contour in the upper half-plane.

$$I = \frac{A}{2\pi} \frac{1}{\mathrm{i}L} \oint_{\text{u.h.p}} \frac{\mathrm{e}^{\mathrm{i}tz}}{z - \mathrm{i}\frac{R}{L}} \, \mathrm{d}z.$$

The only singular point is located in the upper half-plane at $z = \mathrm{i}\frac{R}{L}$. Therefore

$$I = \frac{A}{2\pi} \frac{1}{\mathrm{i}L} 2\pi\mathrm{i} \lim_{z \to \mathrm{i}\frac{R}{L}} \left(z - \mathrm{i}\frac{R}{L} \right) \frac{\mathrm{e}^{\mathrm{i}tz}}{z - \mathrm{i}\frac{R}{L}} = \frac{A}{L} \mathrm{e}^{\mathrm{i}t\left(\mathrm{i}\frac{R}{L}\right)} = \frac{A}{L} \mathrm{e}^{-\frac{R}{L}t}.$$

For $t < 0$, we must close the contour in the lower half-plane. Since there is no singular point in the lower half-plane, the integral is zero. Thus

$$I(t) = \begin{cases} \frac{A}{L} \mathrm{e}^{-\frac{R}{L}t} & t > 0, \\ 0 & t < 0. \end{cases}$$

For those who are familiar with AC circuits, this integral $I(t)$ is the current in a circuit with resistance R and inductance L connected in series under a voltage impulse V. A high pulse in a short duration can be expressed as

$$V(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega,$$

and the impedance of the circuit is $Z = R + \mathrm{i}\omega L$ for the ω component, and the corresponding current is given by $\frac{V}{Z}$. Thus the total current is the integral we have evaluated.

Example 3.5.9. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)}\,\mathrm{d}x.$$

Solution 3.5.9.

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)}\,\mathrm{d}x = \operatorname{Im}\int_{-\infty}^{\infty} \frac{x\mathrm{e}^{\mathrm{i}x}}{(x^2+1)}\,\mathrm{d}x,$$

$$\int_{-\infty}^{\infty} \frac{x\mathrm{e}^{\mathrm{i}x}}{(x^2+1)}\,\mathrm{d}x = \oint_{\mathrm{u.h.p}} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)}\,\mathrm{d}z.$$

There is only one singular point in the upper half-plane located at $z = \mathrm{i}$. So

$$\begin{aligned}\oint_{\mathrm{u.h.p}} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)}\,\mathrm{d}z &= 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}}\left[\frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)}\right] \\ &= 2\pi\mathrm{i}\lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)} = 2\pi\mathrm{i}\frac{\mathrm{i}\mathrm{e}^{-1}}{2\mathrm{i}} = \frac{\pi}{\mathrm{e}}\mathrm{i},\end{aligned}$$

and

$$I = \operatorname{Im}\left(\frac{\pi}{\mathrm{e}}\mathrm{i}\right) = \frac{\pi}{\mathrm{e}}.$$

Example 3.5.10. (a) Show that

$$I = \int_0^{\infty} \frac{\cos bx}{x^2+a^2}\mathrm{d}x = \frac{\pi}{2a}\mathrm{e}^{-ba}, \qquad a > 0, \qquad b > 0.$$

(b) Use the result of (a) to find the value of

$$\int_0^{\infty} \frac{\cos bx}{(x^2+a^2)^2}\,\mathrm{d}x.$$

Solution 3.5.10. (a) Since the integrand is an even function, so

$$\int_0^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x.$$

$$\int_{-\infty}^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \mathrm{Re}\int_{-\infty}^\infty \frac{\mathrm{e}^{\mathrm{i}bx}}{x^2+a^2}\,\mathrm{d}x = \mathrm{Re}\oint_{\mathrm{u.h.p}} \frac{\mathrm{e}^{\mathrm{i}bz}}{z^2+a^2}\,\mathrm{d}z$$

The singular point in the upper half-plane is at $z = \mathrm{i}a$, so

$$\oint_{\mathrm{u.h.p}} \frac{\mathrm{e}^{\mathrm{i}bz}}{z^2+a^2}\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}a}\left[\frac{\mathrm{e}^{\mathrm{i}bz}}{z^2+a^2}\right]$$
$$= 2\pi\mathrm{i}\lim_{z\to\mathrm{i}a}(z-\mathrm{i}a)\frac{\mathrm{e}^{\mathrm{i}bz}}{(z-\mathrm{i}a)(z+\mathrm{i}a)} = 2\pi\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}b(\mathrm{i}a)}}{2a\mathrm{i}} = \frac{\pi}{a}\mathrm{e}^{-ba}.$$

Thus,

$$\int_0^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{Re}\left(\frac{\pi}{a}\mathrm{e}^{-ba}\right) = \frac{\pi}{2a}\mathrm{e}^{-ba}.$$

(b) Taking derivative of both sides with respect to a,

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\pi}{2a}\mathrm{e}^{-ba}\right),$$

we have

$$\int_0^\infty \frac{-2a\cos bx}{(x^2+a^2)^2}\mathrm{d}x = \frac{-\pi}{2a^2}\mathrm{e}^{-ba} + \frac{\pi(-b)}{2a}\mathrm{e}^{-ba}.$$

Therefore

$$\int_0^\infty \frac{\cos bx}{(x^2+a^2)^2}\mathrm{d}x = \frac{\pi}{4a^3}(1+ab)\,\mathrm{e}^{-ba}.$$

3.5.4 Improper Integrals II: Closing the Contour with Rectangular and Pie-shaped Contour

If the integrand does not go to zero fast enough on the infinitely large contour C_R, then the contour cannot be closed with a large semicircle, up or down. For such a case, there may be other types of closed contours that will enable us to eliminate all parts of the integral but the desired portion. However, selecting appropriate contour requires considerable ingenuity. Here we present two additional kinds of contours that are known to be useful.

Rectangular Contour

If the height of the rectangle can be so chosen that the integral along the top side of the rectangle is equal to a constant multiple of the integral along the real axis, then such a contour may be useful for evaluating integrals whose integrand vanishes as the absolute value of the real variable goes to infinity. Generally, integrands containing exponential function or hyperbolic functions are good candidates for this method. Again the method is best illustrated by an example.

Example 3.5.11. Show that

$$I=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x=\frac{\pi}{\sin a\pi},\qquad \text{for}\quad 0<a<1.$$

Solution 3.5.11. First we analytically continue the integrand to the complex plane

$$f(z)=\frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}.$$

The denominator of $f(z)$ is unchanged if z is increased by $2\pi\mathrm{i}$, whereas the numerator changes by a factor of $\mathrm{e}^{a2\pi\mathrm{i}}$. Thus a rectangular contour shown in Fig. 3.11 may be appropriate.

Integrating around the rectangular loop, we have

$$\oint f(z)\,\mathrm{d}z=J_1+J_2+J_3+J_4,$$

where

$$J_1=\int_{L_1}\frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\mathrm{d}z=\int_{-R}^{R}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x,$$

$$J_2=\int_{L_2}\frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z=\int_{0}^{2\pi}\frac{\mathrm{e}^{a(R+\mathrm{i}y)}}{1+\mathrm{e}^{R+\mathrm{i}y}}\mathrm{i}\,\mathrm{d}y,$$

$$J_3=\int_{L_3}\frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z=\int_{R}^{-R}\frac{\mathrm{e}^{a(x+\mathrm{i}2\pi)}}{1+\mathrm{e}^{(x+\mathrm{i}2\pi)}}\,\mathrm{d}x=\mathrm{e}^{\mathrm{i}2\pi a}\int_{R}^{-R}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x,$$

$$J_4=\int_{L_3}\frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z=\int_{2\pi}^{0}\frac{\mathrm{e}^{a(-R+\mathrm{i}y)}}{1+\mathrm{e}^{-R+\mathrm{i}y}}\mathrm{i}\,\mathrm{d}y.$$

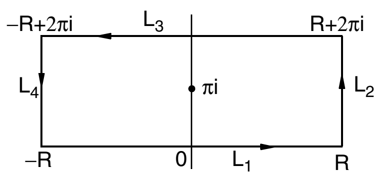

Fig. 3.11. A closed rectangular contour

As $R \to \infty$,

$$\lim_{R\to\infty} J_1 = \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x = I,$$

$$\lim_{R\to\infty} J_3 = \lim_{R\to\infty} \mathrm{e}^{\mathrm{i}2\pi a} \int_{R}^{-R} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x = -\mathrm{e}^{\mathrm{i}2\pi a} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x = -\mathrm{e}^{\mathrm{i}2\pi a} I.$$

Furthermore, since $\left|\mathrm{e}^{a(R+\mathrm{i}y)}\right| = \mathrm{e}^{aR}$ and the minimum value of $\left|1+\mathrm{e}^{R+\mathrm{i}y}\right|$ is $\left|1-\mathrm{e}^{R}\right|$, hence

$$\lim_{R\to\infty} |J_2| \le \lim_{R\to\infty} \left|\frac{\mathrm{e}^{aR}}{1-\mathrm{e}^{R}}\right| 2\pi = \lim_{R\to\infty} \frac{2\pi}{\mathrm{e}^{(1-a)R}} \to 0, \quad \text{since } a < 1.$$

Similarly,

$$\lim_{R\to\infty} |J_4| \le \lim_{R\to\infty} \left|\frac{\mathrm{e}^{-aR}}{1-\mathrm{e}^{-R}}\right| 2\pi = \lim_{R\to\infty} 2\pi\mathrm{e}^{-aR} \to 0, \quad \text{since } a > 0.$$

Therefore

$$\lim_{R\to\infty} \oint \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z = (1-\mathrm{e}^{\mathrm{i}2\pi a})I.$$

Now inside the loop, there is a simple pole at $z = \mathrm{i}\pi$, since

$$1+\mathrm{e}^{z} = 1+\mathrm{e}^{\mathrm{i}\pi} = 1-1 = 0.$$

By the residue theorem, we have

$$\oint \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}\pi}\left[\frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\right] = 2\pi\mathrm{i}\left[\frac{\mathrm{e}^{az}}{(1+\mathrm{e}^{z})'}\right]_{z=\mathrm{i}\pi}$$
$$= 2\pi\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\pi a}}{\mathrm{e}^{\mathrm{i}\pi}} = -2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi a}.$$

Thus,

$$(1-\mathrm{e}^{\mathrm{i}2\pi a})I = -2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi a},$$

so

$$I = \frac{-2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi a}}{1-\mathrm{e}^{\mathrm{i}2\pi a}} = \frac{-2\pi\mathrm{i}}{\mathrm{e}^{-\mathrm{i}\pi a}-\mathrm{e}^{\mathrm{i}\pi a}} = \frac{\pi}{\sin \pi a}.$$

Pie-shaped Contour

If the integral is from 0 to ∞, instead of from $-\infty$ to ∞ and none of the earlier methods is applicable, then a pie-shaped contour may work. In the following example, we will illustrate this method with the evaluation of the Fresnel integrals, which are important in diffraction theory and signal propagation.

Example 3.5.12. Evaluate the Fresnel integrals

$$I_c = \int_0^\infty \cos\left(x^2\right) \mathrm{d}x, \qquad I_s = \int_0^\infty \sin\left(x^2\right) \mathrm{d}x.$$

Solution 3.5.12. The two Fresnel integrals are the real and imaginary parts of the exponential integral,

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = I_c + \mathrm{i}I_s.$$

We integrate the complex function $\mathrm{e}^{\mathrm{i}z^2}$ around the pie-shaped contour shown in Fig. 3.12. Since the function is analytic within the closed contour, the loop integral must be zero,

$$\oint \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = 0.$$

This loop integral naturally divides into three parts. First from 0 to R along the real x axis, then along the path of an arc C_R from R to R'. Finally returning to 0 along the straight radial line with $\theta = \pi/4$.

$$\int_0^R \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x + \int_{C_R} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z + \int_{R'}^0 \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = 0.$$

In the limit of $R \to \infty$, the first integral is what we want to find,

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = \int_0^\infty \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x.$$

On the path of the third integral, with $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $\theta = \pi/4$,

$$\begin{aligned} z^2 &= r^2\mathrm{e}^{\mathrm{i}2\theta} = r^2\mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}r^2, \\ \mathrm{d}z &= \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}r = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{d}r, \end{aligned}$$

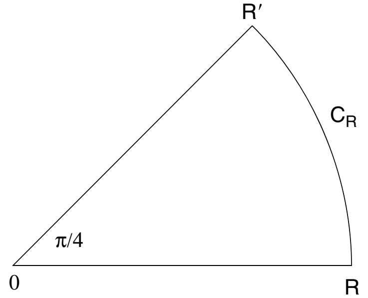

Fig. 3.12. A pie-shaped contour. In the complex plane, R' is at $z\left(R'\right) = R\mathrm{e}^{\mathrm{i}\pi/4}$

so the third integral becomes

$$\int_{R'}^{0} e^{iz^2} dz = \int_{R}^{0} e^{-r^2} e^{i\pi/4} dr = -e^{i\pi/4} \int_{0}^{R} e^{-r^2} dr.$$

We will now show that the second integral along C_R is equal to zero in the limit of $R \to \infty$. On C_R

$$\begin{aligned} z &= Re^{i\theta}, \qquad dz = iRe^{i\theta} d\theta, \\ z^2 &= R^2 e^{i2\theta} = R^2(\cos 2\theta + i \sin 2\theta), \end{aligned}$$

so the second integral can be written as

$$\begin{aligned} \int_{C_R} e^{iz^2} dz = iR \int_0^{\pi/4} & e^{iR^2(\cos 2\theta + i \sin 2\theta)} e^{i\theta} \\ d\theta = iR \int_0^{\pi/4} & e^{i(R^2 \cos 2\theta + \theta)} e^{-R^2 \sin 2\theta} d\theta. \end{aligned}$$

Thus

$$\left| \int_{R}^{R'} e^{iz^2} dz \right| \leq R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta = \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin \phi} d\phi,$$

where $\phi = 2\theta$. According to (3.23) of Jordan's lemma,

$$\int_0^{\pi/2} e^{-R^2 \sin \phi} d\phi \leq \frac{\pi}{2R^2} (1 - e^{-R^2}).$$

Therefore, it goes to zero as $1/R^2$ for $R \to \infty$. With the second integral equal to zero, we are left with

$$\int_0^{\infty} e^{ix^2} dx - e^{i\pi/4} \int_0^{\infty} e^{-r^2} dr = 0. \tag{3.28}$$

Now it is well known that

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

To verify this expression, define

$$I = \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-y^2} dy,$$

so

$$I^2 = \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy.$$

In polar coordinates

$$I^2 = \int_0^\infty \int_0^{\pi/2} \mathrm{e}^{-\rho^2} \rho \, \mathrm{d}\varphi \, \mathrm{d}\rho = \frac{\pi}{2} \int_0^\infty \mathrm{e}^{-r^2} \rho \, \mathrm{d}\rho = \frac{\pi}{4},$$

so $I = \sqrt{\pi}/2$.

It follows from (3.28) that

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = \mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \mathrm{e}^{-r^2} \mathrm{d}r = \mathrm{e}^{\mathrm{i}\pi/4} \frac{\sqrt{\pi}}{2} = \left(\cos \frac{\pi}{4} + \mathrm{i} \sin \frac{\pi}{4}\right) \frac{\sqrt{\pi}}{2}$$
$$= \left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{8}} + \mathrm{i}\sqrt{\frac{\pi}{8}}.$$

Therefore

$$\int_0^\infty \cos\left(x^2\right) \mathrm{d}x = \int_0^\infty \sin\left(x^2\right) \mathrm{d}x = \sqrt{\frac{\pi}{8}}.$$

3.5.5 Integration Along a Branch Cut

Some integrals of multivalued functions can also be evaluated by Cauchy's residue theorem. For example, the integrand of the integral

$$I = \int_0^\infty x^{-\alpha} f(x) \, \mathrm{d}x$$

is multivalued if α is not an integer. In the complex plane, $z^{-\alpha}$ is multivalued because with z expressed as

$$z = r\mathrm{e}^{\mathrm{i}(\theta + n2\pi)},$$

where n is an integer, $z^{-\alpha}$ becomes

$$z^{-\alpha} = \mathrm{e}^{-\alpha \ln z} = \mathrm{e}^{-\alpha(\ln r + \mathrm{i}\theta + \mathrm{i}n2\pi)}.$$

It is seen that $z^{-\alpha}$ is a multivalued function. For instance, with $\alpha = 1/3$,

$$z^{-\frac{1}{3}} = \begin{cases} \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} & n = 0 \\ \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} \mathrm{e}^{\mathrm{i}2\pi/3} = \left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} & n = 1 \\ \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} \mathrm{e}^{\mathrm{i}4\pi/3} = \left(-\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} & n = 2. \end{cases}$$

To define $z^{-\alpha}$ as a single valued function, the angle θ must be restricted in an interval of 2π by a branch cut. If we choose the branch cut along the positive x axis, then our real integral is an integral along the top of the branch cut. Very often the problem can be solved with a closed contour as shown in Fig. 3.13, in which the entire branch cut is excluded from the interior of the contour. Again let us illustrate the method with an example.

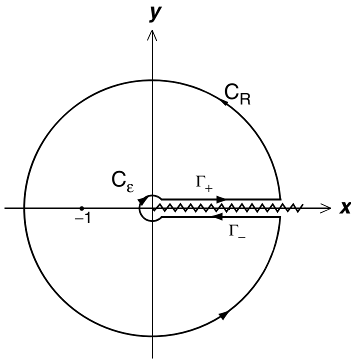

Fig. 3.13. A contour that excludes the branch cut along the positive x axis

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Example 3.5.13. Evaluate the integral

$$I = \int_0^\infty \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x, \qquad 0 < \alpha < 1.$$

Solution 3.5.13. Consider the contour integral

$$\oint \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z$$

around the closed contour shown in Fig. 3.13. Since the branch point at $z = 0$ and the entire branch cut are excluded, the only singular point inside this contour is at $z = -1$. Therefore

$$\oint \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}\left[\frac{z^{-\alpha}}{1+z}\right] = 2\pi\mathrm{i}\,(-1)^{-\alpha} = 2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(-\alpha)} = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}}.$$

This integral consists of four parts

$$\oint \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = \int_{\Gamma_+} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z + \int_{C_R} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z + \int_{\Gamma_-} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z + \int_{C_\epsilon} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z.$$

The first integral is along the top of the branch cut with $\theta = 0$, the second integral is along the outer large circle with radius R, the third integral is along the bottom of the branch cut with $\theta = 2\pi$, and the fourth integral is along the inner small circle with radius ϵ.

With $z = r\mathrm{e}^{\mathrm{i}\theta}$, it is clear that when $\theta = 0$ and $\theta = 2\pi$, r is the same as x. Therefore

$$\int_{\Gamma_+} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = \int_\epsilon^R \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x,$$

$$\int_{\Gamma_-} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = \int_R^\epsilon \frac{x^{-\alpha}\mathrm{e}^{\mathrm{i}2\pi(-\alpha)}}{1+x\mathrm{e}^{i2\pi}}\mathrm{e}^{\mathrm{i}2\pi}\,\mathrm{d}x = -\mathrm{e}^{-\mathrm{i}2\pi\alpha}\int_\epsilon^R \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x.$$

On C_R, $z = R\mathrm{e}^{\mathrm{i}\theta}$,

$$\left| \int_{C_R} \frac{z^{-\alpha}}{1+z} \, \mathrm{d}z \right| \leq \left| \frac{R^{-\alpha}}{1-R} 2\pi R \right|,$$

where $R^{-\alpha}$ is maximum of the numerator, $1 - R$ is the minimum of denominator, and $2\pi R$ is the length of C_R. As $R \to \infty$,

$$\left| \frac{R^{-\alpha}}{1-R} 2\pi R \right| \sim R^{-\alpha} \to 0, \quad \text{since} \quad \alpha > 0.$$

Similarly, on C_ϵ, $z = \epsilon \mathrm{e}^{\mathrm{i}\theta}$,

$$\left| \int_{C_\epsilon} \frac{z^{-\alpha}}{1+z} \, \mathrm{d}z \right| \leq \frac{\epsilon^{-\alpha}}{1-\epsilon} 2\pi\epsilon.$$

As $\epsilon \to 0$,

$$\frac{\epsilon^{-\alpha}}{1-\epsilon} 2\pi\epsilon \sim \epsilon^{1-\alpha} \to 0, \quad \text{since} \quad \alpha < 1.$$

On taking the limit $R \to \infty$ and $\epsilon \to 0$, we are left with

$$\oint \frac{z^{-\alpha}}{1+z} \, \mathrm{d}z = \int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x - \mathrm{e}^{-\mathrm{i}2\pi\alpha} \int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}}.$$

Thus

$$\left(1 - \mathrm{e}^{-\mathrm{i}2\pi\alpha}\right) \int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}},$$

and

$$\int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}\left(1 - \mathrm{e}^{-\mathrm{i}2\pi\alpha}\right)} = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}} = \frac{\pi}{\sin \pi\alpha}.$$

3.5.6 Principal Value and Indented Path Integrals

Sometimes we have to deal with integrals $\int f(x) \, \mathrm{d}x$ whose integrand becomes infinite at a point $x = x_0$ in the range of integration

$$\lim_{x \to x_0} f(x) = \infty.$$

In order to make sense of this kind of integral, we define the principal value integral as

$$P \int_{-R}^{R} f(x) \, \mathrm{d}x = \lim_{\varepsilon \to 0} \left[\int_{-R}^{x_0 - \varepsilon} f(x) \, \mathrm{d}x + \int_{x_0 + \varepsilon}^{R} f(x) \, \mathrm{d}x \right].$$

It is a way to avoid the singularity. One integrates to within a small distance ε of the singularity in question, skips over the singularity, and begins integrating again at a distance ε beyond the singularity.

When evaluating the integral using the residue theorem, we are not allowed to have a singularity on the contour, however, with principal value integrals we can accommodate simple poles on the contour by deforming the contour so as to avoid the poles.

The principal value integral

$$P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

can be evaluated by the theorem of residue for a function $f(z)$ that satisfies the asymptotic conditions that we have discussed. That is, either $zf(z) \to 0$ as $z \to \infty$, or $f(z) = \mathrm{e}^{\mathrm{i}mz} g(z)$ and $g(z) \to 0$ as $z \to \infty$. Let us first assume that $f(z)$ has one simple pole on the real axis at $z = x_0$ and is analytic everywhere else. In this case, it is clear that the closed contour integral around the indented path shown in Fig. 3.14 is equal to zero

$$\oint f(z)\,\mathrm{d}z = 0,$$

since the only singular point is outside the contour.

The integral can be written as

$$\oint f(z)\,\mathrm{d}z = \int_{-R}^{x_0-\varepsilon} f(x)\,\mathrm{d}x + \int_{C_\varepsilon} f(z)\,\mathrm{d}z + \int_{x_0+\varepsilon}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z.$$

In the limit of $R \to \infty$, with $f(z)$ satisfying the specified conditions, we have shown

$$\int_{C_R} f(z)\,\mathrm{d}z = 0.$$

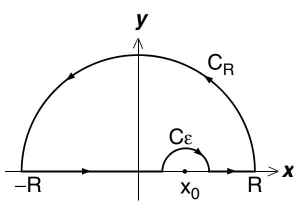

Fig. 3.14. The closed contour consists of a large semicircle C_R in the upper half-plane of radius R, the line segments from $-R$ to $x_0 - \varepsilon$ and from $x_0 + \varepsilon$ to R along the real axis, and a small semicircle C_ε of radius ϵ above the singular point x_0

Furthermore, the two line integrals along the x axis become the principal value integral as $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{x_0-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_0+\varepsilon}^{\infty} f(x)\,\mathrm{d}x\right] = P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

Since $f(z)$ has a simple pole at $z = x_0$, so in the immediate neighborhood of x_0, the Laurent series of $f(z)$ has the form

$$f(z) = \frac{a_{-1}}{z - x_0} + \sum_{n=0}^{\infty} a_n (z - x_0)^n.$$

On the semicircle C_ε around x_0,

$$z - x_0 = \varepsilon \mathrm{e}^{\mathrm{i}\theta}, \qquad \mathrm{d}z = \mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta,$$

where ε is the radius of the semicircle. The integral around C_ε can thus be written as

$$\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \int_{\pi}^{0}\left(\frac{a_{-1}}{\varepsilon \mathrm{e}^{\mathrm{i}\theta}} + \sum_{n=0}^{\infty} a_n \left(\varepsilon \mathrm{e}^{\mathrm{i}\theta}\right)^n\right)\mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta.$$

On taking the limit $\varepsilon \to 0$, every term vanishes except the first. Therefore

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \int_{\pi}^{0} a_{-1}\mathrm{i}\mathrm{d}\theta = -\mathrm{i}\pi a_{-1} = -\mathrm{i}\pi \mathrm{Res}_{z=x_0}[f(z)].$$

It follows that in the limit $R \to \infty$ and $\varepsilon \to 0$

$$\oint f(z)\,\mathrm{d}z = P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - \mathrm{i}\pi \mathrm{Res}[f(z)] = 0.$$

Therefore

$$P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \mathrm{i}\pi \mathrm{Res}[f(z)].$$

Note that to avoid the singular point, we can just as well go below it instead above. For the semicircle below the x axis, the direction of integration is counterclockwise

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \int_{\pi}^{2\pi} a_{-1}\mathrm{i}\mathrm{d}\theta = \mathrm{i}\pi a_{-1} = \mathrm{i}\pi \mathrm{Res}_{z=x_0}[f(z)].$$

However, in that case, the singular point is inside the closed contour, and the loop integral is equal to $2\pi\mathrm{i}$ times the residue at $z = x_0$. So we have

$$\oint f(z)\,\mathrm{d}z = P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \mathrm{i}\pi \mathrm{Res}_{z=x_0}[f(z)] = 2\pi\mathrm{i}\,\mathrm{Res}_{z=x_0}[f(z)].$$

Not surprisingly we get the same result

$$P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\,\mathrm{Res}_{z=x_0}\left[f(z)\right] - \mathrm{i}\pi\mathrm{Res}_{z=x_0}\left[f(z)\right] = \mathrm{i}\pi\mathrm{Res}_{z=x_0}\left[f(z)\right].$$

Now if $f(z)$ has more than one pole on the real axis, (all of them first-order), furthermore, it has other singularities in the upper half-plane, (not necessary first-order), then by the same argument one can show that

$$\begin{aligned} P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi i &\left(\sum \text{residues on } x \text{ axis}\right) \\ &+2\pi\mathrm{i}\left(\sum \text{residues in upper half-plane}\right). \end{aligned}$$

Example 3.5.14. Find the principal value of

$$P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x$$

and use the result to show

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

Solution 3.5.14. The only singular point is at $x = 0$, therefore

$$P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = \pi\mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{\mathrm{e}^{\mathrm{i}z}}{z}\right] = \pi\mathrm{i}\left[\frac{\mathrm{e}^{\mathrm{i}z}}{z'}\right]_{z=0} = \pi\mathrm{i}.$$

Since

$$P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = P\left[\int_{-\infty}^{\infty} \frac{\cos x}{x}\,\mathrm{d}x + \mathrm{i}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x\right],$$

therefore,

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \mathrm{Im}\left(P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x\right) = \pi.$$

We note that $x = 0$ is actually a removable singularity of $\sin x/x$, since as $x \to 0$, $\sin x/x = 1$. This means that ε, instead of approaching zero, can be set equal to exactly zero. Therefore the principal value of the integral is the integral itself

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

It is instructive to check this result in the following way. Since $\sin x/x$ is continuous at $x = 0$, if we move the path of integration an infinitesimal amount at $x = 0$, the value of the integral will not be changed. So let the path go through an infinitesimally small semicircle C_ε on top of the point at $x = 0$. Let us call the indented path as the path from $-\infty$ to ε along x axis, followed by C_ε and then continue from ε to ∞ along the x axis. Then

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \int_{\text{Indented}} \frac{\sin z}{z}\,\mathrm{d}z.$$

Now let us make use of the identity

$$\sin z = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right),$$

so

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2\mathrm{i}}\int_{\text{Indented}} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z - \frac{1}{2\mathrm{i}}\int_{\text{Indented}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\,\mathrm{d}z.$$

For the first integral in the right-hand side, we can close the contour with an infinitely large semicircle C_R^+ in the upper half-plane, as shown in Fig. 3.15a. Since the singular point is outside the closed contour, so the contour integral vanishes

$$\int_{\text{Indented}} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z = \oint_{u,h.p} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z = 0.$$

For the second indented path integral, we cannot close the contour in the upper half-plane because of $\mathrm{e}^{-\mathrm{i}z}$, so we have to close the contour in the lower half-plane with C_R^-, as shown in Fig. 3.15b. In this case, the singular point at $z = 0$ is inside the contour, therefore

$$\int_{\text{Indented}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\,\mathrm{d}z = \oint_{\text{l.h.p}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\,\mathrm{d}z = -2\pi\mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{\mathrm{e}^{-\mathrm{i}z}}{z}\right] = -2\pi\mathrm{i}.$$

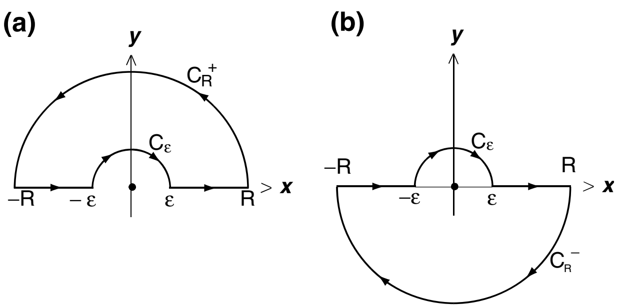

Fig. 3.15. (**a**) The indented path from $-R$ to R is closed by a large semicircle C_R^+ in the upper half-plane. (**b**) The same indented path from $-R$ to R is closed by a large semicircle C_R^- in the lower half-plane

The minus sign accounts for the clockwise direction. It follows that:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = \frac{1}{2\mathrm{i}} 0 - \frac{1}{2\mathrm{i}} (-2\pi \mathrm{i}) = \pi,$$

which is the same as we obtained before.

Exercises

1. Expand $f(z) = \frac{z-1}{z+1}$ in a Taylor's series (a) around $z = 0$ and (b) around the point $z = 1$. Determine the radius of convergence of each series.

   Ans:

$$\text{(a)} f(z) = -1 + 2z - 2z^2 + 2z^2 - \cdots \quad |z| < 1$$
$$\text{(b)} f(z) = \frac{1}{2}(z-1) - \frac{1}{4}(z-1)^2 + \frac{1}{8}(z-1)^3 - \frac{1}{16}(z-1)^4 + \cdots |z-1| < 2.$$

2. Find the Taylor series expansion about the origin and the radius of convergence for

$$\text{(a)} \quad f(z) = \sin z, \qquad \text{(b)} \quad f(z) = \cos z,$$
$$\text{(c)} \quad f(z) = \mathrm{e}^z, \qquad \text{(d)} \quad f(z) = \frac{1}{(1-z)^m}.$$

   Ans:

$$\text{(a)} \quad \sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots \quad \text{all} \quad z,$$
$$\text{(b)} \quad \cos z = 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \cdots \quad \text{all} \quad z,$$
$$\text{(c)} \ \mathrm{e}^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots \quad \text{all} \quad z,$$
$$\text{(d)} \quad \frac{1}{(1-z)^m} = 1 + mz + \frac{m(m+1)}{2}z^2 + \frac{m(m+1)(m+2)}{3!}z^3 + \cdots |z| < 1.$$

3. Find the Taylor series expansion of

$$f(z) = \ln z$$

   around $z = 1$ by noting that

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln z = \frac{1}{z}.$$

Ans:
$\ln z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \cdots \qquad |z-1| < 1.$

4. Expand

$$f(z) = \frac{1}{(z+1)(z+2)}$$

in a Taylor's series (a) around $z = 0$ and (b) around the point $z = 2$. Determine the radius of convergence of each series.

Ans:

$$\text{(a)}\ \ f(z) = \frac{1}{2} - \frac{3}{4}z + \frac{7}{8}z^2 - \frac{15}{16}z^3 + \cdots \qquad |z| < 1.$$

$$\text{(b)}\ \ f(z) = \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{3^2} - \frac{1}{4^2}\right)(z-2)$$

$$+ \left(\frac{1}{3^3} - \frac{1}{4^3}\right)(z-2)^2 - \cdots \qquad |z-2| < 3.$$

5. Without obtaining the series, determine the radius of convergence of each of the following expansions:

$$\text{(a)}\ \tan^{-1} z \quad \text{around } z = 1,$$

$$\text{(b)}\ \frac{1}{e^z - 1} \quad \text{around } \ z = 4i,$$

$$\text{(c)}\ \frac{x}{x^2 + 2x + 10} \quad \text{around } \ x = 0.$$

Ans: (a) $\sqrt{2}$, (b) $2\pi - 4$, (c) $\sqrt{10}$.

6. Find the Laurent series for

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

in the region of

$$\text{(a)}\ |z| < 1,\ \ \text{(b)}\ \ 1 < |z| < 2,\ \ \text{(c)}\ \ 0 < |z-1| < 1,$$

$$\text{(d)}\ \ 2 < |z|,\ \text{(e)}\ \ |z-1| > 1,\ \ \text{(f)}\ \ 0 < |z-2| < 1.$$

Ans:

$$\text{(a)}\ \ f(z) = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \cdots$$

$$\text{(b)}\ \ f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3.$$

$$\text{(c)}\ \ f(z) = -\frac{1}{(z-1)} - 1 - (z-1) - (z-1)^2 - \cdots$$

(d) $f(z) = \cdots + \frac{15}{z^5} + \frac{7}{z^4} + \frac{3}{z^3} + \frac{1}{z^2}.$

(e) $f(z) = \cdots + \frac{1}{(z-1)^4} + \frac{1}{(z-1)^3} + \frac{1}{(z-1)^2}.$

(f) $f(z) = \frac{1}{z-2} - 1 + (z-2) - (z-2)^2 + (z-2)^3 - \cdots$

7. Expand

$$f(z) = \frac{1}{z^2(z-\mathrm{i})}$$

in two different Laurent expansions around $z = \mathrm{i}$ and tell where each converges.

Ans:
$f(z) = -\frac{1}{z-\mathrm{i}} - 2\mathrm{i} + 3(z-\mathrm{i}) - 4\mathrm{i}(z-\mathrm{i})^2 + \cdots \qquad 0 < |z-\mathrm{i}| < 1$
$f(z) = \cdots \frac{4\mathrm{i}}{(z-\mathrm{i})^6} - \frac{3}{(z-\mathrm{i})^5} - \frac{2\mathrm{i}}{(z-\mathrm{i})^4} + \frac{1}{(z-\mathrm{i})^3} \qquad |z-\mathrm{i}| > 1.$

8. Find the values of $\oint_C f(z)\,\mathrm{d}z$, where C is the circle $|z| = 3$, for the following functions:

(a) $f(z) = \frac{1}{z(z+2)}$, (b) $f(z) = \frac{z+2}{z(z+1)}$, (c) $f(z) = \frac{z}{(z+1)(z+2)}$,
(d) $f(z) = \frac{1}{z(z+1)^2}$, (e) $f(z) = \frac{1}{(z+1)^2}$, (f) $f(z) = \frac{1}{z(z+1)(z+4)}$

by expanding them in an appropriate Laurent series $f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n$ and using $a_{-1} = \frac{1}{2\pi\mathrm{i}} \oint_C f(z)\mathrm{d}z$.

Ans: (a) 0, (b) $2\pi\mathrm{i}$, (c) $2\pi\mathrm{i}$, (d) 0, (e) 0, (f) $-\mathrm{i}\pi/6$.

9. Find the residue of

$$f(z) = \frac{z}{z^2+1}$$

(a) at $z = \mathrm{i}$ and (b) at $z = -\mathrm{i}$.
Ans: (a) 1/2; (b) 1/2.

10. Find the residue of

$$f(z) = \frac{z+1}{z^2(z-2)}$$

(a) at $z = 0$ and (b) at $z = 2$.
Ans: (a) $-3/4$; (b) 3/4.

11. Find the residue of

$$f(z)=\frac{z}{z^2+2z+5}$$

at each of its poles.
Ans: $r(-1+2\mathrm{i})=(2+\mathrm{i})/4; \quad r(-1-2\mathrm{i})=(2-\mathrm{i})/4$

12. What is the residue of

$$f(z)=\frac{1}{(z+1)^3}$$

at $z=-1$?
Ans: 0.

13. What is the residue of

$$f(z)=\tan z$$

at $z=\pi/2$?
Ans: -1

14. What is the residue of

$$f(z)=\frac{1}{z-\sin z}$$

at $z=0$?
Ans: 3/10

15. Use the theory of residue to evaluate $\oint\limits_C f(z)\mathrm{d}z$ if C is the circle $|z|=4$ for each of the following functions:

$$\text{(a)}\ \frac{z}{z^2-1}, \qquad \text{(b)}\ \frac{z+1}{z^2(z+2)}, \qquad \text{(c)}\ \frac{1}{z(z-2)^3},$$

$$\text{(d)}\ \frac{1}{z^2+z+1} \qquad \text{(e)}\ \frac{1}{z(z^2+6z+4)}.$$

Ans. (a) $2\pi\mathrm{i}$, (b) 0, (c) 0, (d) 0, (e) $(5-3\sqrt{5})\mathrm{i}\pi/20$.

16. Show that

$$\oint_C \frac{1}{(z^{100}+1)(z-4)}\,\mathrm{d}z=\frac{-2\pi\mathrm{i}}{4^{100}+1},$$

if C is the circle $|z|=3$.
Hint: First find the value of the integral along $|z|=5$, then do the integration along $|z|=5$ and $|z|=3$ with a cut between them.

17. Use the theory of residue to evaluate the following definite integrals

$$\text{(a)} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2+\cos\theta},$$

$$\text{(b)} \int_0^{2\pi} \frac{\cos 3\theta \mathrm{d}\theta}{5-4\cos\theta},$$

$$\text{(c)} \int_0^{\pi} \frac{\cos 2\theta \mathrm{d}\theta}{1-2a\cos\theta+a^2} \quad \text{where} \quad (-1<a<1),$$

$$\text{(d)} \int_0^{\pi} \sin^{2n}\theta \mathrm{d}\theta \quad \text{where} \quad n=1,2,\ldots.$$

Ans. (a) $2\pi/\sqrt{3}$, (b) $\pi/12$, (c) $\pi a^2/\left(1-a^2\right)$, (d) $\pi\,(2n)!/\left(2^{2n}(n!)^2\right)$.

18. Show that

$$\text{(a)} \int_0^{\infty} \frac{\mathrm{d}x}{x^2+1} = \frac{\pi}{2},$$

$$\text{(b)} \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1} \mathrm{d}x = \sqrt{2}\pi$$

$$\text{(c)} \int_0^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{4},$$

$$\text{(d)} \int_0^{\infty} \frac{ab}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x = \frac{\pi}{2(a+b)}.$$

19. Evaluate

$$\text{(a)} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}3x}}{x-2\mathrm{i}} \mathrm{d}x,$$

$$\text{(b)} \int_0^{\infty} \frac{\cos kx}{x^2+1} \mathrm{d}x,$$

$$\text{(c)} \int_{-\infty}^{\infty} \frac{\cos mx}{(x-a)^2+b^2} \mathrm{d}x,$$

$$\text{(d)} \int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x.$$

Ans. (a) $2\pi\mathrm{i}/\mathrm{e}^6$, (b) $\frac{\pi}{2}\,\mathrm{e}^{-|k|}$, (c) $\frac{\pi}{b}\,\mathrm{e}^{-mb}\cos ma$, (d) $\frac{\pi}{a^2-b^2}\left(\frac{\mathrm{e}^{-bm}}{b}-\frac{\mathrm{e}^{-am}}{a}\right)$.

20. Use a rectangular contour to show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{\mathrm{e}^{-x}+\mathrm{e}^{x}} \mathrm{d}x = \frac{\pi}{\mathrm{e}^{m\pi/2}+\mathrm{e}^{-m\pi/2}}.$$

21. Use the "integration along the branch cut" method to show that

$$\int_0^{\infty} \frac{x^{1/3}}{(1+x)^2} \mathrm{d}x = \frac{2\pi}{3\sqrt{3}}.$$

22. Use a pie-shaped contour with $\theta = 2\pi/3$ to show that

$$\int_0^\infty \frac{1}{x^3+1}\,\mathrm{d}x = \frac{2\sqrt{3}\pi}{9}.$$

23. Find the principal value of the following

$$P\int_{-\infty}^{\infty} \frac{1}{(x+1)\left(x^2+2\right)}\,\mathrm{d}x.$$

Ans. $\sqrt{2}\pi/6$.

24. Show that

$$\frac{1-\mathrm{e}^{2\mathrm{i}z}}{z^2}$$

has a simple pole at $z=0$. Find the principal value of

$$P\int_{-\infty}^{\infty} \frac{1-\mathrm{e}^{2\mathrm{i}x}}{x^2}\,\mathrm{d}x.$$

Use the result to show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

Ans. 2π.

Part II

Determinants and Matrices

4

Determinants

Determinants are powerful tools for solving systems of linear equations and they are indispensable in the development of matrix theory. Most readers probably already possess the knowledge of evaluating second- and third-order determinants. After a systematic review, we introduce the formal definition of a nth order determinant through the Levi-Civita symbol. All properties of determinants can be derived from this definition.

4.1 Systems of Linear Equations

4.1.1 Solution of Two Linear Equations

Suppose we wish to solve for x and y from the system of 2×2 linear equations (2 equations and 2 unknowns)

$$a_1 x + b_1 y = d_1, \tag{4.1}$$

$$a_2 x + b_2 y = d_2, \tag{4.2}$$

where a_1, a_2, b_1, b_2, d_1, and d_2 are known constants. We can multiply (4.1) by b_2 and (4.2) by b_1, and then take the difference. In so doing, y is eliminated, and we are left with

$$(b_2 a_1 - b_1 a_2)x = b_2 d_1 - b_1 d_2,$$

therefore

$$x = \frac{d_1 b_2 - d_2 b_1}{a_1 b_2 - a_2 b_1}, \tag{4.3}$$

where we have written $b_2 a_1$ as $a_1 b_2$, since the order is immaterial in the product of two numbers. It turns out that if we use the following notation, it is much easier to generalize this process to larger systems of $n \times n$ equations

$$a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{4.4}$$

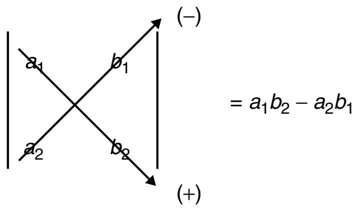

Fig. 4.1. A schematic diagram for a second-order determinant

The 2×2 square array of the four elements on the right-hand side of this equation is called a second-order determinant. Its meaning is just that its value is equal to the left-hand side of this equation. Explicitly, the value of a second-order determinant is defined as the difference between the two products of the diagonal elements as shown in the schematic diagram (Fig. 4.1).

With determinants, (4.3) can be written as

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{4.5}$$

and with a similar procedure one can easily show that

$$y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{4.6}$$

Example 4.1.1. Find the solution of

$$\begin{aligned} 2x - 3y &= -4, \\ 6x - 2y &= 2. \end{aligned}$$

Solution 4.1.1.

$$x = \frac{\begin{vmatrix} -4 & -3 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix}} = \frac{8+6}{-4+18} = 1,$$

$$y = \frac{\begin{vmatrix} 2 & -4 \\ 6 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix}} = \frac{4+24}{-4+18} = 2.$$

4.1.2 Properties of Second-Order Determinants

There are many general properties of determinants that will be discussed in later sections. At this moment we want to list a few which we need in the following discussion of third-order determinant. For a second-order determinant, these properties are almost self-evident from its definition. Although they are generally valid for nth order determinant, at this point we only need them to be valid for second-order determinant to continue our discussion:

1. If the rows and columns are interchanged, the determinant is unaltered,
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{4.7}$$
2. If two columns (or two rows) are interchanged, the determinant changes sign,
$$\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1) = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{4.8}$$
3. If each element in a column (or in a row) is multiplied by m, the determinant is multiplied by m,
$$\begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix} = ma_1b_2 - ma_2b_1 = m(a_1b_2 - a_2b_1) = m\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$
4. If each element of a column (or of a row) is sum of two terms, the determinant equals the sum of the two corresponding determinants,
$$\begin{vmatrix} (a_1+c_1) & b_1 \\ (a_2+c_2) & b_2 \end{vmatrix} = (a_1+c_1)b_2 - (a_2+c_2)b_1 = a_1b_2 - a_2b_1 + c_1b_2 - c_2b_1$$
$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}.$$

4.1.3 Solution of Three Linear Equations

Now suppose we want to solve a system of three equations

$$a_1x + b_1y + c_1z = d_1, \tag{4.9}$$

$$a_2x + b_2y + c_2z = d_2, \tag{4.10}$$

$$a_3x + b_3y + c_3z = d_3. \tag{4.11}$$

First we can solve for y and z in terms of x. Writing (4.10) and (4.11) as

$$b_2y + c_2z = d_2 - a_2x,$$

$$b_3y + c_3z = d_3 - a_3x,$$

then in analogy to (4.5) and (4.6), we can express y and z as

$$y = \frac{\begin{vmatrix} (d_2 - a_2 x) & c_2 \\ (d_3 - a_3 x) & c_2 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}, \tag{4.12}$$

$$z = \frac{\begin{vmatrix} b_2 & (d_2 - a_2 x) \\ b_3 & (d_3 - a_3 x) \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}. \tag{4.13}$$

Substituting these two expressions into (4.9) and then multiplying the entire equation by

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$

we have

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} x + b_1 \begin{vmatrix} (d_2 - a_2 x) & c_2 \\ (d_3 - a_3 x) & c_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & (d_2 - a_2 x) \\ b_3 & (d_3 - a_3 x) \end{vmatrix} = d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}. \tag{4.14}$$

By properties 3 and 4, this equation becomes

$$\begin{aligned} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} x &+ b_1 \left\{ \begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} x \right\} \\ &+ c_1 \left\{ \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} - \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} x \right\} = d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}. \end{aligned} \tag{4.15}$$

It follows:

$$Dx = N_x, \tag{4.16}$$

where

$$N_x = d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix}, \tag{4.17}$$

and

$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}. \tag{4.18}$$

Expanding the second-order determinants, (4.18) leads to

$$D = a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 a_3 c_2 - c_1 b_2 a_3 + c_1 b_3 a_2. \tag{4.19}$$

To express these six terms in a more systematic way, we introduce a third-order determinant as a short hand notation for (4.19)

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{4.20}$$

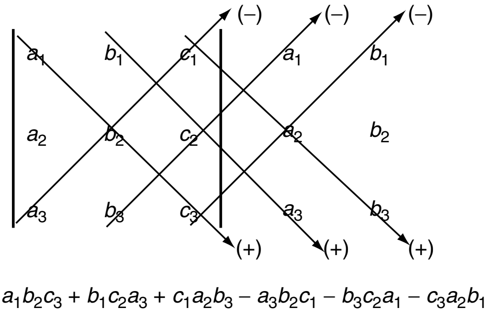

Fig. 4.2. A schematic diagram for a third-order determinant

A useful device for evaluating a third-order determinant is as follows. We write down the determinant column by column, after the third column, we repeat the first, then the second column, creating a 3×5 array of numbers. We can form a product of three elements along each of the three diagonals going from upper left to lower right. These products carry a positive sign. Similarly, three products can be formed along the diagonals from lower left to upper right. These three latter products carry a minus sign. The value of the determinant is equal to the sum of these six terms. This is shown in the diagram (Fig. 4.2).

This is seen to be exactly equal to the six terms in (4.19).

Using the determinant notation, one can easily show that N_x in (4.17) is equal to

$$N_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}. \tag{4.21}$$

Therefore

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Similarly we can define

$$N_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad N_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

and show

$$y = \frac{N_y}{D}, \qquad z = \frac{N_z}{D}.$$

The determinant in the denominator D is called the determinant of the coefficients. It is simply formed with the array of the coefficients on the left-hand sides of (4.9)–(4.11). To find the numerator determinant N_x, start with D, erase the x coefficients a_1, a_2, and a_3, and replace them with the constants d_1, d_2, and d_3 from the right-hand sides of the equations. Similarly we replace the y coefficients in D with the constant terms to find N_y, and the z coefficients in D with the constants to find N_z.

Example 4.1.2. Find the solution of

$$\begin{aligned} 3x + 2y + z &= 11, \\ 2x + 3y + z &= 13, \\ x + y + 4z &= 12. \end{aligned}$$

Solution 4.1.2.

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 36 + 2 + 2 - 3 - 3 - 16 = 18,$$

$$N_x = \begin{vmatrix} 11 & 2 & 1 \\ 13 & 3 & 1 \\ 12 & 1 & 4 \end{vmatrix} = 132 + 24 + 13 - 36 - 11 - 104 = 18,$$

$$N_y = \begin{vmatrix} 3 & 11 & 1 \\ 2 & 13 & 1 \\ 1 & 12 & 4 \end{vmatrix} = 156 + 11 + 24 - 13 - 36 - 88 = 54,$$

$$N_z = \begin{vmatrix} 3 & 2 & 11 \\ 2 & 3 & 13 \\ 1 & 1 & 12 \end{vmatrix} = 108 + 26 + 22 - 33 - 39 - 48 = 36.$$

Thus

$$x = \frac{18}{18} = 1, \quad y = \frac{54}{18} = 3, \quad z = \frac{36}{18} = 2.$$

Clearly, with determinant notation, the results can be given in a systematic way. While this procedure is still valid for systems of more than three equations, as we shall see in the section on Cramer's rule, but the diagonal scheme of expanding the determinants shown in this section is generally correct only for determinants of second- and third-orders. For determinants of higher order, we must pay attention to the formal definition of determinants.

4.2 General Definition of Determinants

4.2.1 Notations

Before we present the general definition of an arbitrary order determinant, let us write the third-order determinant in a more systematic way. Equations (4.19) and (4.20) can be written in the following form:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} a_i b_j c_k, \tag{4.22}$$

where

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \\ &\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1, \\ &\varepsilon_{ijk} = 0 \quad \text{for all others.} \end{aligned} \tag{4.23}$$

Writing out term by term the right-hand side of (4.22), one can readily verify that the six nonvanishing terms are exactly the same as in (4.19).

In order to generalize this definition for a nth order determinant, let us examine the triple sum more closely. First we note that $\varepsilon_{ijk} = 0$ if any two of the three indices i, j, k are equal, e.g., $\varepsilon_{112} = 0$, $\varepsilon_{333} = 0$. Eliminating those terms, (4.22) is a particular linear combination of six products, each product contains one and only one element from each row and from each column. Each product carries either a positive or a negative sign. The arrangements of (i, j, k) in the positive products are either in the normal order of $(1, 2, 3)$, or are the results of an even number of interchanges between two adjacent numbers of the normal order. Those in the negative products are the results of an odd number of interchanges in the normal order. For example, it takes two interchanges to get $(2, 3, 1)$ from $(1, 2, 3)$ [123 (interchange 12)→213 (interchange 13)→231], and $a_2 b_3 c_1$ is positive ($\varepsilon_{231} = 1$); it takes only one interchange to get $(1, 3, 2)$ from $(1, 2, 3)$ [123 (interchange 23)→132], and $a_1 b_3 c_2$ is negative ($\varepsilon_{132} = -1$). The diagram (Fig. 4.3) can help us to find out the value of ε_{ijk} quickly. If a set of indices goes in the clockwise direction, it gives a positive one (+1), if it goes in the counterclockwise direction, it gives a negative one (−1).

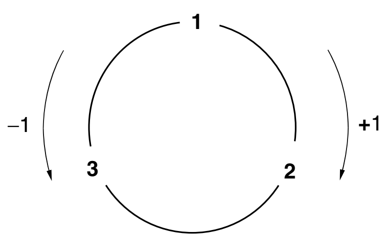

Fig. 4.3. Levi-Civita symbol ε_{ijk} where i, j, k take the value of $1, 2$, or 3. If the set of indices goes clockwise, $\varepsilon_{ijk} = +1$, if counterclockwise, $\varepsilon_{ijk} = -1$

These properties are characterized by the Levi-Civita symbol $\varepsilon_{i_1 i_2 \cdots i_n}$, which is defined as follows:

$$\varepsilon_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & \text{if} \quad (i_1, i_2, \ldots, i_n) \text{ is an even permutation} \\ & \quad\quad \text{of the normal order } (1, 2, \ldots, n) \\ -1 & \text{if} \quad (i_1, i_2, \ldots, i_n) \text{ is an odd permutation} \\ & \quad\quad \text{of the normal order } (1, 2, \ldots, n) \\ 0 & \quad\quad \text{if any index is repeated.} \end{cases}$$

An even permutation means that an even number of pairwise interchanges of adjacent numbers is needed to obtain the given permutation from the normal order, and an odd permutation is associated with an odd number of pairwise interchanges. As we have shown, $(2, 3, 1)$ is an even permutation, and $(1, 3, 2)$ is an odd permutation.

An easy way to determine whether a given permutation is even or odd is to write out the normal order and write the permutation directly below it. Then connect corresponding numbers in these two arrangements with line segments, and count the number of intersections between pairs of these lines. If the number of intersections is even, then the given permutation is even. If the number of intersections is odd, then the permutation is odd. For example, to find the permutation $(2, 3, 4, 1)$, we write out the normal order and permutation in the diagram (Fig. 4.4, we call it "permutation diagram"):

There are three intersections. Therefore the permutation is odd and $\varepsilon_{2341} = -1$. The reason this scheme is valid is because of the following. Starting with the smallest number that is not directly below the same number, an exchange of this number with the number to its left will eliminate one intersection. In the earlier example, after the interchange between 1 and 4, only two intersections remain. Clearly two more interchanges will eliminate all intersections and return the permutation to the normal order. Thus three intersections indicate three interchanges are needed. Therefore the permutation is odd.

When we count the number of intersections, we are counting the intersections of pairs of lines. Therefore one should avoid to have more than two lines intersecting at a point. The lines joining the corresponding numbers need not to be straight lines.

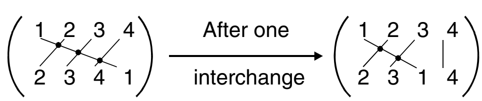

Fig. 4.4. Permutation diagram. The permutation is written directly below the normal order. The number of intersections between pairs of lines connecting the corresponding numbers is equal to the number of interchanges needed to obtain the permutation from the normal order. This diagram shows that one intersection point represents one interchange between two adjacent members

Example 4.2.1. What is the value of the the Levi-Civita symbol $\varepsilon_{1357246}$?

Solution 4.2.1. There are six intersections in Fig. 4.5, therefore the permutation is even and $\varepsilon_{1357246} = 1$.

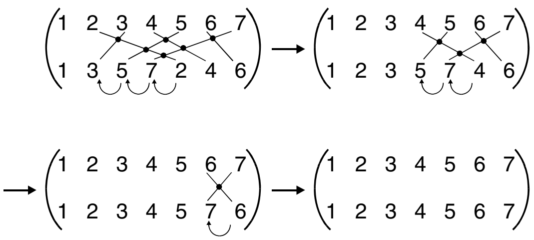

Fig. 4.5. In this diagram, six intersections represent that six interchanges are needed to obtain the permutation 1357246 from the normal order 1234567

4.2.2 Definition of a nth Order Determinant

In discussing a general nth order determinant, it is convenient to use the double-subscript notation. Each element of the determinant is represented by the symbol a_{ij}. The subscripts ij indicate that it is the element at ith row and jth column. With this notation, $a_1b_2c_3$ becomes $a_{11}a_{22}a_{33}$; $a_2b_3c_1$ becomes $a_{21}a_{32}a_{13}$, and $a_ib_jc_k$ becomes $a_{i1}a_{j2}a_{k3}$. The determinant itself is denoted by a variety of symbols. The following notations are all equivalent:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}| = |A| = \det|A| = D_n. \tag{4.24}$$

The value of the determinant is given by

$$D_n = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \varepsilon_{i_1 i_2 \cdots i_n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}. \tag{4.25}$$

This equation is the formal definition of a nth order determinant. Clearly, for $n = 3$, it reduces to (4.22). Note that for a nth order determinant, there are $n!$ possible products because i_1 can take one of n values, i_2 cannot repeat i_1, so it can take only one of $n-1$ values, and so on. We can think of evaluating a determinant in terms of three steps. (1) Take $n!$ products of n elements such that in each product there is one and only one element from each row and one and only one element from each column. (2) Attach a positive (+) sign to the product if the row subscripts are an even permutation of the column subscripts, and a minus sign $(-)$ if an odd perturbation. (3) Sum over $n!$ products with these signs.

Stated in this way, it is clear that the definition of a determinant is symmetrical between the rows and columns. The determinant (4.25) can just as well be written as

$$D_n = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \varepsilon_{i_1 i_2 \cdots i_n} a_{1 i_1} a_{2 i_2} \cdots a_{n i_n} . \tag{4.26}$$

It follows that any theorem about the determinant which involves the rows is also true for the columns, and vice-versa.

Another property that is clear from this definition is this. If any two rows are interchanged, the determinant changes sign. First it is easy to show that if the two rows are adjacent to each other, this is the case. This follows from the fact that an interchange of two adjacent rows corresponds to an interchange of two adjacent row indices in the Levi-Civita symbol. It changes an even permutation into an odd permutation, and vice versa. Therefore it introduces a minus sign to all the products.

Now suppose the row indices i and j are not adjacent to each other and there are n indices between them:

$$i \;\; a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_n \;\; j \, .$$

To bring j to the left requires $n+1$ adjacent interchanges leading to

$$j \;\; i \;\; a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_n \, .$$

Now bringing i to the right requires n adjacent interchanges leading to

$$j \;\; a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_n \;\; i \, .$$

Therefore all together there are $2n+1$ number of adjacent interchanges leading to the interchange of i and j. Since $2n+1$ is an odd integer, this brings in an overall minus sign.

Example 4.2.2. Let

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

use (4.25) to (a) expand this second-order determinant, (b) show explicitly that the interchange of the two rows changes its sign.

Solution 4.2.2. (a) According to (4.25)

$$D_2 = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2},$$

$$\begin{aligned}
i_1 = 1,\ i_2 = 1: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{11} a_{11} a_{12} \\
i_1 = 1,\ i_2 = 2: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{12} a_{11} a_{22} \\
i_1 = 2,\ i_2 = 1: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{21} a_{21} a_{12} \\
i_1 = 2,\ i_2 = 2: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{22} a_{21} a_{22}.
\end{aligned}$$

Since $\varepsilon_{11} = 0,\ \varepsilon_{12} = 1,\ \varepsilon_{21} = -1,\ \varepsilon_{22} = 0$, the double sum gives the second-order determinant as

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = a_{11} a_{22} - a_{21} a_{12}.$$

(b) To express the interchange of two rows, we can simply replace $a_{i_1 1} a_{i_2 2}$ in the double sum with $a_{i_2 1} a_{i_1 2}$ (i_1 and i_2 are interchanged), thus

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_2 1} a_{i_1 2}.$$

Since i_1 and i_2 are running indices, we can rename i_1 as j_2 and i_2 as j_1, so

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_2 1} a_{i_1 2} = \sum_{j_2=1}^{2} \sum_{j_1=1}^{2} \varepsilon_{j_2 j_1} a_{j_1 1} a_{j_2 2} = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \varepsilon_{j_2 j_1} a_{j_1 1} a_{j_2 2}.$$

The last expression is identical with that of the original determinant except the indices of the Levi-Civita symbol are interchanged.

$$\begin{aligned}
\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} &= \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \varepsilon_{j_2 j_1} a_{j_1 1} a_{j_2 2} \\
&= \varepsilon_{11} a_{11} a_{12} + \varepsilon_{21} a_{11} a_{22} + \varepsilon_{12} a_{21} a_{12} + \varepsilon_{22} a_{21} a_{22} \\
&= -a_{11} a_{22} + a_{21} a_{12} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.
\end{aligned}$$

This result can, of course, be obtained by inspection. We have taken the risk of stating the obvious. Hopefully, this step by step approach will remove any uneasy feeling of working with indices.

4.2.3 Minors, Cofactors

Let us return to (4.18), written in the double-subscript notation this equation becomes

$$\begin{aligned}
D_3 &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \qquad (4.27)
\end{aligned}$$

where we have interchanged the two columns of the last second-order determinant of (4.18) and changed the sign. It is seen that

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

is the second-order determinant formed by removing the first row and first column from the original third-order determinant D_3. We call it M_{11} the minor complementary to a_{11}. In general, the minor M_{ij} complementary to a_{ij} is defined as the $(n-1)$th order determinant formed by deleting the ith row and the jth column from the original nth order determinant D_n. The cofactor C_{ij} is defined as $(-1)^{i+j}M_{ij}$.

Example 4.2.3. Find the value of the minors M_{11}, M_{23} and the cofactors C_{11}, C_{23} of the determinant

$$D_4 = \begin{vmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & 2 & 3 & 1 \end{vmatrix}.$$

Solution 4.2.3.

$$M_{11} = \begin{vmatrix} * & * & * & * \\ * & 2 & 5 & 0 \\ * & 0 & -2 & 2 \\ * & 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 0 \\ 0 & -2 & 2 \\ 2 & 3 & 1 \end{vmatrix}; \quad M_{23} = \begin{vmatrix} 2 & -1 & * & 3 \\ * & * & * & * \\ 1 & 0 & * & 2 \\ 4 & 2 & * & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & 2 & 1 \end{vmatrix}.$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 5 & 0 \\ 0 & -2 & 2 \\ 2 & 3 & 1 \end{vmatrix}; \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & 2 & 1 \end{vmatrix}.$$

4.2.4 Laplacian Development of Determinants by a Row (or a Column)

With these notations, (4.27) becomes

$$D_3 = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} M_{1j} \tag{4.28}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \sum_{k=1}^{3} a_{1k} C_{1k}. \tag{4.29}$$

This is known as the Laplace development of the third-order determinant on elements of the first row. It turns out this is not limited to the third-order determinant. It is a fundamental theorem that determinants of any order can be evaluated by a Laplace development on any row or column

$$D_n = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any } i, \tag{4.30}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any } j. \tag{4.31}$$

The proof may be given by induction and is based on the definition of the determinant. According to (4.25), a determinant is the sum of all the $n!$ products which are formed by taking exactly one element from each row and each column and multiplying by 1 or -1 in accordance with the Levi-Civita rule.

Now the minor M_{ij} of a nth order determinant is a $(n-1)$th determinant. It is a sum of $(n-1)!$ products. Each product has one element from each row and each column except the ith row and jth column. It is then clear that $\sum_{j=1}^{n} k_{ij} a_{ij} M_{ij}$ is a sum of $n(n-1)! = n!$ products, and each product is formed with exactly one element from each row and each column. It follows that, with the appropriate choice of k_{ij}, the determinant can be written in a row expansion

$$D_n = \sum_{j=1}^{n} k_{ij} a_{ij} M_{ij} \tag{4.32}$$

or in a column expansion

$$D_n = \sum_{i=1}^{n} k_{ij} a_{ij} M_{ij}. \tag{4.33}$$

The Laplace development will follow if we can show:

$$k_{ij} = (-1)^{i+k}.$$

First let us consider all the terms in (4.25) containing a_{11}. In these terms $i_1 = 1$. We note that if $(1, i_2, i_3, \ldots, i_n)$ is an even (or odd) permutation of $(1, 2, 3, \ldots, n)$, it means $(i_2, i_3, \ldots, i_n)$ is an even (or odd) permutation of $(2, 3, \ldots, n)$. The number of intersections in the following two "permutation diagrams" are obviously the same:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}; \qquad \begin{pmatrix} 2 & 3 & \cdots & n \\ i_2 & i_3 & \cdots & i_n \end{pmatrix},$$

therefore

$$\varepsilon_{1 i_2 \cdots i_n} = \varepsilon_{i_2 \cdots i_n}.$$

So terms containing a_{11} sum to

$$\sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \varepsilon_{1 i_2 \cdots i_n} a_{11} a_{i_2 2} \cdots a_{i_n n} = a_{11} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \varepsilon_{i_2 \cdots i_n} a_{i_2 2} \cdots a_{i_n n},$$

which is simply $a_{11}M_{11}$, where M_{11} is the minor of a_{11}. On the other hand, according to (4.32), all the terms containing a_{11} sum to $k_{11}a_{11}M_{11}$. Therefore

$$k_{11} = +1.$$

Next consider the terms in (4.25) which contain a particular element a_{ij}. If we interchange the ith row with the one above it, the determinant changes sign. If we move the row up in this way $(i-1)$ times, the ith row will have moved up into the first row, and the order of the other rows is not changed. The process will change the sign of the determinant $(i-1)$ times. In a similar way, we can move the jth column to the first column without change the order of the other columns. Then the element a_{ij} will be in the top left corner of the determinant, in the place of a_{11}, and the sign of the determinant has change $(i-1+j-1)$ times. That is

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j-2} \begin{vmatrix} a_{ij} & a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{1j} & a_{11} & a_{12} & \cdots & a_{1n} \\ a_{2j} & a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{nj} & a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}.$$

In the rearranged determinant, a_{ij} is in the place of a_{11}, thus the sum of all the terms containing a_{ij} is equal to $a_{ij}M_{ij}$. But there is a factor $(-1)^{i+j-2}$ in front of the rearranged determinant. Therefore the terms containing a_{ij} in the right-hand side of the equation sum to $(-1)^{i+j-2}a_{ij}M_{ij}$. On the other hand, according to (4.32), all the terms containing a_{ij} in the determinant of the left-hand side of the equation sum to $k_{ij}a_{ij}M_{ij}$. Therefore,

$$k_{ij} = (-1)^{i+j-2} = (-1)^{i+j}. \tag{4.34}$$

This completes the proof of the Laplace development, which is very important in both theory and computation of determinants. It is useful to keep in mind that k_{ij} forms a checkboard pattern:

$$\begin{vmatrix} +1 & -1 & +1 & & & \\ -1 & 1 & -1 & & & \\ +1 & -1 & +1 & & & \\ & & & \cdots & & \\ & & & & +1 & -1 \\ & & & & -1 & +1 \end{vmatrix}.$$

Example 4.2.4. Find the value of the determinant

$$D_3 = \begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix}$$

by (a) a Laplace development on the first row; (b) a Laplace development on the second row; (c) a Laplace development on the first column.

Solution 4.2.4.

(*a*)

$$\begin{aligned} D_3 &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 3\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\ &= 3(4+3) + 2(2+12) + 2(1-8) = 35. \end{aligned}$$

(b)

$$\begin{aligned} D_3 &= -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23} \\ &= -1\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} \\ &= -(-4-2) + 2(6-8) + 3(3+8) = 35. \end{aligned}$$

(c)

$$\begin{aligned} D_3 &= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \\ &= 3\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} \\ &= 3(4+3) - (-4-2) + 4(6-4) = 35. \end{aligned}$$

Example 4.2.5. Find the value of the *triangular determinant*

$$D_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix}.$$

Solution 4.2.5.

$$D_n = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

4.3 Properties of Determinants

By mathematical induction, we can now show that properties 1 to 4 of second-order determinants are generally valid for nth order determinants. Based on the fact that it is true for $(n-1)$th order determinants, we will show that it must also be true for nth order determinants. All properties of the determinant can be derived directly from its definition of (4.25). However, in this section, we will demonstrate them with Laplace expansions.

1. The value of the determinant remains the same if rows and columns are interchanged.

Let the Laplace expansion of D_n on elements of the first row be

$$D_n = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}. \tag{4.35}$$

Let D_n^{T} (known as the transpose of D_n) be the nth order determinant formed by interchanging rows and columns of the determinant D_n. The Laplace expansion of D_n^{T} on elements of the first column (which are elements of the first row of D_n) is then given by

$$D_n^{\mathrm{T}} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}^{\mathrm{T}}, \tag{4.36}$$

where M_{1j}^{T} is the minor complement to a_{1j}, and is equal to the determinant M_{1j} with rows and columns interchanged. In the case of $n=3$, the minors are second-order determinants. By (4.7), $M_{1j}^{\mathrm{T}} = M_{1j}$. Therefore $D_3 = D_3^{\mathrm{T}}$. This process can be carried out, one step at a time, to any n. Therefore we conclude

$$D_n = D_n^{\mathrm{T}}. \tag{4.37}$$

2. The determinant changes sign if any two columns (or any two rows) are interchanged.

First we will verify this property for the third-order determinant D_3. Let E_3 be the determinant obtained by interchanging two columns of D_3. Suppose column k is not one of those exchanged. Using Laplace development to expand D_3 and E_3 by their kth column, we have

$$D_3 = \sum_{i=1}^{3} (-1)^{i+k} a_{ik} M_{ik}; \tag{4.38}$$

$$E_3 = \sum_{i=1}^{3} (-1)^{i+k} a_{ik} M'_{ik}, \tag{4.39}$$

where M'_{ik} is a second-order determinant and is equal to M_{ik} with the two columns interchanged. By (4.8), $M'_{ik} = -M_{ik}$. Hence $E_3 = -D_3$. Now by mathematical induction, we assume this property holds for $(n-1)$th order determinants. The same procedure will show that this property also holds for determinants of nth order.

This property is called antisymmetric property. It is frequently used in quantum mechanics in the construction of an antisymmetric many particle wave functions.

3. If each element in a column (or in a row) is multiplied by a constant m, the determinant is multiplied by m.

This property follows directly from the Laplacian expansion. If the ith column is multiplied by m, this property can be shown in the following way:

$$\begin{vmatrix} a_{11} & \cdots & ma_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & ma_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & ma_{ni} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} ma_{ji}C_{ji} = m\sum_{j=1}^{n} a_{ji}C_{ji}$$
$$= m \begin{vmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix}. \tag{4.40}$$

4. If each element in a column (or in a row) is a sum of two terms, the determinant equals the sum of the two corresponding determinants.

If the ith column is a sum of two terms, we can expand the determinant on elements of the ith column

$$\begin{vmatrix} a_{11} & \cdots & a_{1i}+b_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i}+b_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni}+b_{ni} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n}(a_{ji}+b_{ji})C_{ji} = \sum_{j=1}^{n} a_{ji}C_{ji} + \sum_{j=1}^{n} b_{ji}C_{ji}$$
$$= \begin{vmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & b_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & b_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & b_{ni} & \cdots & a_{nn} \end{vmatrix}. \tag{4.41}$$

From these four properties, one can derive many others. For example:

5. If two columns (or two rows) are the same, the determinant is zero.

This follows from the antisymmetric property. If we exchange the two identical columns, the determinant will obviously remain the same. Yet the antisymmetric property requires the determinant to change sign. The only number that is equal to its negative self is zero. Therefore the determinant must be zero.

6. The value of a determinant is unchanged if a multiple of one column is added to another column (or if a multiple of one row is added to another row).

   Without loss of generality, this property can be expressed as follows:

$$\begin{vmatrix} a_{11}+ma_{12} & a_{12} & \cdots & a_{1n} \\ a_{21}+ma_{22} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}+ma_{n2} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} ma_{12} & a_{12} & \cdots & a_{1n} \\ ma_{22} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ ma_{n2} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$
$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + m \begin{vmatrix} a_{12} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n2} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{4.42}$$

   The first equal sign is by property 4, the second equal sign is because of property 3, and the last equal sign is due to property 5.

Example 4.3.1. Show that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Solution 4.3.1.

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & (bc+a^2) \\ 1 & b & (ac+ab) \\ 1 & c & (ab+ac) \end{vmatrix} = \begin{vmatrix} 1 & a & (bc+a^2+ba) \\ 1 & b & (ac+ab+b^2) \\ 1 & c & (ab+ac+bc) \end{vmatrix}$$
$$= \begin{vmatrix} 1 & a & (bc+a^2+ba+ca) \\ 1 & b & (ac+ab+b^2+cb) \\ 1 & c & (ab+ac+bc+c^2) \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & (bc+ba+ca) \\ 1 & b & (ac+ab+cb) \\ 1 & c & (ab+ac+bc) \end{vmatrix}$$
$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + (ab+bc+ca) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

First we multiply each element of the second column by a and add to the third column. For the second equal sign, we multiply the second column by b and add to the third column. Do the same thing except multiplying by c for the third equal sign. The fourth equal sign is due to property 4. The fifth equal sign is due to property 3. And lastly, the determinant with two identical column vanishes.

Example 4.3.2. Evaluate the determinant

$$D_n = \begin{vmatrix} 1+a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & 1+a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & 1+a_3 & \cdots & a_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_1 & a_2 & a_3 & \cdots & 1+a_n \end{vmatrix}.$$

Solution 4.3.2. Adding column 2, column 3, all the way to column n to column 1, we have

$$D_n = \begin{vmatrix} 1+a_1+a_2+a_3+\cdots+a_n & a_2 & a_3 & \cdots & a_n \\ 1+a_1+a_2+a_3+\cdots+a_n & 1+a_2 & a_3 & \cdots & a_n \\ 1+a_1+a_2+a_3+\cdots+a_n & a_2 & 1+a_3 & \cdots & a_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1+a_1+a_2+a_3+\cdots+a_n & a_2 & a_3 & \cdots & 1+a_n \end{vmatrix}$$

$$= (1+a_1+a_2+a_3+\cdots+a_n) \begin{vmatrix} 1 & a_2 & a_3 & \cdots & a_n \\ 1 & 1+a_2 & a_3 & \cdots & a_n \\ 1 & a_2 & 1+a_3 & \cdots & a_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & a_2 & a_3 & \cdots & 1+a_n \end{vmatrix}.$$

Multiplying row 1 by -1 and add it to row 2, and then add it to row 3, and so on

$$D_n = (1+a_1+a_2+a_3+\cdots+a_n) \begin{vmatrix} 1 & a_2 & a_3 & \cdots & a_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= (1+a_1+a_2+a_3+\cdots+a_n).$$

Example 4.3.3. Evaluate the following determinants (known as *Vandermonde determinant*):

$$\text{(a)}\quad D_3 = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}, \qquad \text{(b)}\quad D_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

Solution 4.3.3. (a) Method I.

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & (x_2-x_1) & (x_2^2-x_1^2) \\ 0 & (x_3-x_1) & (x_3^2-x_1^2) \end{vmatrix} = (x_2-x_1)(x_3-x_1)\begin{vmatrix} 1 & (x_2+x_1) \\ 1 & (x_3+x_1) \end{vmatrix}$$
$$= (x_2-x_1)(x_3-x_1)(x_3-x_2).$$

Method II. D_3 is a polynomial in x_1 and it vanishes when $x_1 = x_2$, since then the first two rows are the same. Hence it is divisible by $(x_1 - x_2)$. Similarly, it is divisible by $(x_2 - x_3)$ and $(x_3 - x_1)$. Therefore

$$D_3 = k\,(x_1-x_2)\,(x_1-x_3)\,(x_2-x_3)\,.$$

Furthermore, since D_3 is of degree 3 in x_1, x_2, x_3, k must be a constant. The coefficient of the term $x_2x_3^2$ in this expression is $k(-1)(-1)^2$. On the other hand, the diagonal product of the D_3 is $+x_2x_3^2$. Comparing them shows that $k(-1)(-1)^2 = 1$. Therefore $k = -1$ and

$$D_3 = -\,(x_1-x_2)\,(x_1-x_3)\,(x_2-x_3) = (x_2-x_1)(x_3-x_1)(x_3-x_2).$$

(b) With the same reason as in Method II of (a),

$$D_n = k(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)(x_2-x_3)\cdots(x_2-x_n)\cdots(x_{n-1}-x_n).$$

The coefficient of the term $x_2x_3^2\cdots x_n^{n-1}$ in this expression is $k(-1)(-1)^2\cdots(-1)^{n-1}$. Compare this with the diagonal product of D_n, we have

$$1 = k(-1)(-1)^2\cdots(-1)^{n-1} = k(-1)^{1+2+3+\cdots+(n-1)}.$$

Since

$$1+2+3+\cdots+(n-1) = \frac{1}{2}n(n-1),$$

therefore

$$D_n = (-1)^{n(n-1)/2}(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)(x_2-x_3)\cdots(x_2-x_n)\cdots(x_{n-1}-x_n).$$

Example 4.3.4. Pivotal Condensation. Show that

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}.$$

Clearly, a_{11} must be nonzero. If it is zero, then the first row (or first column) must be exchanged with another row (or another column), so that $a_{11} \neq 0$.

Solution 4.3.4.

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
&= \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & a_{11}a_{12} & a_{11}a_{13} \\ a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{vmatrix} \\
&= \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & (a_{11}a_{12} - a_{11}a_{12}) & (a_{11}a_{13} - a_{11}a_{13}) \\ a_{21} & (a_{11}a_{22} - a_{21}a_{12}) & (a_{11}a_{23} - a_{21}a_{13}) \\ a_{31} & (a_{11}a_{32} - a_{31}a_{12}) & (a_{11}a_{33} - a_{31}a_{13}) \end{vmatrix} \\
&= \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & (a_{11}a_{22} - a_{21}a_{12}) & (a_{11}a_{23} - a_{21}a_{13}) \\ a_{31} & (a_{11}a_{32} - a_{31}a_{12}) & (a_{11}a_{33} - a_{31}a_{13}) \end{vmatrix} \\
&= \frac{1}{a_{11}} \begin{vmatrix} (a_{11}a_{22} - a_{21}a_{12}) & (a_{11}a_{23} - a_{21}a_{13}) \\ (a_{11}a_{32} - a_{31}a_{12}) & (a_{11}a_{33} - a_{31}a_{13}) \end{vmatrix} \\
&= \frac{1}{a_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ & \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}.
\end{aligned}
$$

This method can be applied to reduce a nth order determinant to a $(n-1)$th order determinant and is known as pivotal condensation. It may not offer any advantage for hand calculation, but it is useful in evaluating determinants with computers.

4.4 Cramer's Rule

4.4.1 Nonhomogeneous Systems

Suppose we have a set of n equations and n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\
\cdots\cdots\cdots\cdots &= \cdot \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n.
\end{aligned} \tag{4.43}
$$

The constants $d_1, d_2, \ldots, d_n$ on the right-hand side are known as nonhomogeneous terms. If they are not all equal to zero, the set of equations is known as a nonhomogeneous system. The problem is to find $x_1, x_2, \ldots, x_n$ to satisfy this set of equations. We will see by using the properties of determinants, this set of equations can be readily solved for any n.

Forming the determinant of the coefficients and then multiplying by x_1, with the help of property 3 we have

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}x_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

We multiply the second column of the right-hand side determinant by x_2 and add it to the first column, and then multiply the third column by x_3 and add it to the first column and so on. According to property 6, the determinant is unchanged

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n & a_{22} & \cdots & a_{2n} \\ \cdots & \cdot & \cdots & \cdot \\ a_{n1}x_1 + a_{n2}x_2 \cdots + a_{nn}x_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Replacing the first column of the right-hand side determinant with the constants of the right-hand side of (4.43), we obtain

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} d_1 & a_{12} & \cdots & a_{1n} \\ d_2 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ d_3 & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Clearly if we multiply the determinant of the coefficients by x_2, we can analyze the second column of the determinant in the same way. In general

$$x_i D_n = N_i, \qquad 1 \leq i \leq n, \tag{4.44}$$

where D_n is the determinant of the coefficients

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and N_i is the determinant obtained by replacing the ith column of D_n by the nonhomogeneous terms

$$N_i = \begin{vmatrix} a_{11} & \cdots & a_{1i-1} & d_1 & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i-1} & d_2 & a_{2i+1} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni-1} & d_n & a_{ni+1} & \cdots & a_{nn} \end{vmatrix}. \tag{4.45}$$

Thus if the determinant of the coefficients is not zero, the system has a unique solution

$$x_i = \frac{N_i}{D_n}, \quad 1 \leq i \leq n. \tag{4.46}$$

This procedure is known as Cramer's rule. For the special cases of $n = 2$ and $n = 3$, the results are, of course, identical to what we derived in the first section. Cramer's rule is very important in the development of the theory of determinants and matrices. However, to use it for solving a set of equations with large n, it is not very practical. Either because the amount of computations is so large and/or because the demand of numerical accuracy is so high with this method, even with high speed computers it may not be possible to carry out such calculations. There are other techniques to solve that kind of problems, such as the Gauss-Jordan elimination method which we will discuss in the chapter on matrix theory.

4.4.2 Homogeneous Systems

Now if $d_1, d_2, \ldots, d_n$ in the right-hand side of (4.43) are all zero, that is

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots &= \cdot \\
a_{n1}x_1 + a_{n2}x_2 \cdots + a_{nn}x_n &= 0,
\end{aligned}
$$

the set of equations is known as a homogeneous system. In this case, all N_i's in (4.45) are equal to zero. If $D_n \neq 0$, then the only solution by (4.46) is a trivial one, namely $x_1 = x_2 = \cdots = x_n = 0$. On the other hand, if D_n is equal to zero, then it is clear from (4.44), x_i do not have to be zero. Hence a homogeneous system can have a nontrivial solution only if the coefficient determinant is equal to zero. Conversely, one can show that if

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\cdot & \cdot & \cdots & \cdot \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} = 0, \tag{4.47}
$$

then there is always a nontrivial solution of the homogeneous equations. For a 2×2 system, the existence of a solution can be shown by direct calculation. Then one can show by mathematical induction that the statement is true for any $n \times n$ system.

This simple fact has many important applications.

Example 4.4.1. For what values of λ do the equations

$$
\begin{aligned}
3x + 2y &= \lambda x, \\
4x + 5y &= \lambda y
\end{aligned}
$$

have a solution other than $x = y = 0$?

Solution 4.4.1. Moving the right-hand side to the left gives the homogeneous system

$$\begin{aligned}(3-\lambda)x + 2y &= 0,\\ 4x + (5-\lambda)y &= 0.\end{aligned}$$

For a nontrivial solution, the coefficient determinant must vanish:

$$\begin{vmatrix} 3-\lambda & 2 \\ 4 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 7 = (\lambda-1)(\lambda-7) = 0.$$

Thus the system has a nontrivial solution if and only if $\lambda = 1$ or $\lambda = 7$.

4.5 Block Diagonal Determinants

Frequently we encounter determinants with many zero elements and the nonzero elements which form square blocks along the diagonal. For example the following fifth-order determinant is a block diagonal determinant:

$$D_5 = |A| = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ * & * & a_{33} & a_{34} & a_{35} \\ * & * & a_{43} & a_{44} & a_{45} \\ * & * & a_{53} & a_{54} & a_{55} \end{vmatrix}.$$

In this section, we will show that

$$D_5 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix},$$

regardless the values the elements $*$ assume.

By definition

$$D_5 = \sum_{i_1=1}^{5} \sum_{i_2=1}^{5} \sum_{i_3=1}^{5} \sum_{i_4=1}^{5} \sum_{i_5=1}^{5} \varepsilon_{i_1 i_2 i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5}.$$

Since $a_{13} = a_{14} = a_{15} = a_{23} = a_{24} = a_{25} = 0$, all terms containing these elements can be excluded from the summation. Thus

$$D_5 = \sum_{i_1=1}^{5} \sum_{i_2=1}^{5} \sum_{i_3=3}^{5} \sum_{i_4=3}^{5} \sum_{i_5=3}^{5} \varepsilon_{i_1 i_2 i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5}.$$

Furthermore, the summation over i_1 and i_2 can be written as from 1 to 2, since $3, 4$, and 5 are taken up by i_3, i_4, or i_5, and the Levi-Civita symbol is equal to zero if any index is repeated. Hence

$$D_5 = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_3=3}^{5} \sum_{i_4=3}^{5} \sum_{i_5=3}^{5} \varepsilon_{i_1 i_2 i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5}.$$

Under these circumstances, the permutation of i_1, i_2, i_3, i_4, i_5 can be separated into two permutations as schematically shown later:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i_1 = 1,2 & i_2 = 1,2 & i_3 = 3,4,5 & i_4 = 3,4,5 & i_5 = 3,4,5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ i_1 & i_2 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ i_3 & i_4 & i_5 \end{pmatrix}.$$

The entire permutation is even if the two separated permutations are both even or both odd. The permutation is odd if one of the separated permutations is even and the other is odd. Therefore

$$\varepsilon_{i_1 i_2 i_3 i_4 i_5} = \varepsilon_{i_1 i_2} \cdot \varepsilon_{i_3 i_4 i_5}.$$

It follows

$$\begin{aligned} D_5 &= \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_3=3}^{5} \sum_{i_4=3}^{5} \sum_{i_5=3}^{5} \varepsilon_{i_1 i_2} \cdot \varepsilon_{i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5} \\ &= \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} \cdot \sum_{i_3=3}^{5} \sum_{i_4=3}^{5} \sum_{i_5=3}^{5} \varepsilon_{i_3 i_4 i_5} a_{i_3 3} a_{i_4 4} a_{i_5 5} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix}. \end{aligned}$$

When the blocks are along the "antidiagonal" line, we can evaluate the determinant in a similar way, except we should be careful about its sign. For example,

$$\begin{vmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & * & * \\ a_{41} & a_{42} & * & * \end{vmatrix} = \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}, \tag{4.48}$$

and

$$\begin{vmatrix} 0 & 0 & 0 & a_{14} & a_{15} & a_{16} \\ 0 & 0 & 0 & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & * & * & * \\ a_{51} & a_{52} & a_{53} & * & * & * \\ a_{61} & a_{62} & a_{63} & * & * & * \end{vmatrix} = - \begin{vmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{vmatrix} \cdot \begin{vmatrix} a_{14} & a_{15} & a_{16} \\ a_{24} & a_{25} & a_{26} \\ a_{34} & a_{35} & a_{36} \end{vmatrix}. \tag{4.49}$$

We can establish the result of (4.48) by changing it to a block diagonal determinant with an even number of interchanges between two rows. However, we need an odd number of interchanges between two rows to change (4.49) into a block diagonal determinant, therefore a minus sign.

Solution 4.5.1. *Example 4.5.1.* Evaluate

$$D_5 = \begin{vmatrix} 0 & 2 & 0 & 7 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}.$$

Solution 4.5.2.

$$D_5 = \begin{vmatrix} 0 & 2 & 0 & 7 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} \to (\text{Row } 4 - \text{Row } 2) = \begin{vmatrix} 0 & 2 & 0 & 7 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 5 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -2 \cdot 1 = -2.$$

4.6 Laplacian Developments by Complementary Minors

(This section can be skipped in the first reading.)

The Laplace expansion of D_3 by the elements of the third column is

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

The three second-order determinants are minors complementary to their respective elements. It is also useful to think that the three elements a_{13}, a_{23}, a_{33} are complementary to their respective minors. Obviously the expansion can be written as

$$D_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} a_{33} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} a_{23} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13}. \tag{4.50}$$

In this way, it is seen that the determinant D_3 is equal to the sum of the signed products of all the second-order minors contained in the first two columns, each multiplied by its complementary element. In fact, any determinant D_n, even for $n > 3$, can be expanded in the same way, except the complementary element is of course another complementary minor. For example, for a 4th order determinant

$$D_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \tag{4.51}$$

six second-order minors can be formed from the first two columns. They are

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix}, \quad \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}.$$

Let us expand D_4 in terms of these six minors. First expanding D_4 by its first column, then expanding the four minors by their first columns, we have

$$D_4 = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}, \tag{4.52}$$

where

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{22} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - a_{32} \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} + a_{42} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}$$

$$C_{21} = - \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + a_{32} \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} - a_{42} \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix}$$

$$C_{31} = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{12} \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} - a_{22} \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} + a_{42} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$$

$$C_{41} = - \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} + a_{22} \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} - a_{32} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}.$$

Putting these cofactors back into (4.52) and collecting terms, we have

$$D_4 = (a_{11}a_{22} - a_{21}a_{12})\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - (a_{11}a_{32} - a_{31}a_{12})\begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix}$$
$$+(a_{11}a_{41} - a_{41}a_{12})\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} + (a_{21}a_{32} - a_{31}a_{22})\begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix}$$
$$-(a_{21}a_{42} - a_{41}a_{22})\begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} + (a_{31}a_{42} - a_{41}a_{32})\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}. \quad (4.53)$$

Clearly,

$$D_4 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix}$$
$$- \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}. \quad (4.54)$$

If D_4 is a block diagonal determinant,

$$D_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{vmatrix},$$

then only the first term in (4.54) is nonzero, therefore

$$D_4 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix},$$

in agreement with the result derived in the last section.

If we adopt the following notation

$$A_{i_1i_2,j_1j_2} = \begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} \\ a_{i_2j_1} & a_{i_2j_2} \end{vmatrix}$$

and $M_{i_1i_2,j_1j_2}$ as the complementary minor to $A_{i_1i_2,j_1j_2}$, the determinant D_4 in (4.51) can be expanded in terms of the minors formed by the elements of any two columns,

$$D_4 = \sum_{i_1=1}^{3} \sum_{i_2>i_1}^{4} (-1)^{i_1+i_2+j_1+j_2} A_{i_1i_2,j_1j_2} M_{i_1i_2,j_1j_2}. \quad (4.55)$$

With $j_1 = 1,\ j_2 = 2$, it can be readily verified that (4.55) is, term by term, equal to (4.54). The proof of (4.55) goes the same way as in the Laplacian expansion by a row. First (4.55) is a linear combination of 4! products, each product has one element from each row and one from each column. The coefficients are either $+1$ or -1, depending on whether an even or odd number of interchange are needed to move i_1 to the first row, i_2 to the second row, and j_1 to the first column, j_2 to the second column, without changing the order of the rest of the elements. Obviously, the determinant can also be expanded in terms of the minors formed from any number of rows.

For a nth order determinant D_n, one can expand it in a similar way, not only in terms of second-order minors but also in terms of kth order minors with $k < n$. Of course, for $k = n - 1$, it reduces to the regular Laplacian development by a column. Following the same procedure of expanding D_4, one can show that

$$D_n = \sum_{(i)} (-1)^{i_1+i_2+\cdots+i_k+j_1+j_2+\cdots+j_k} A_{i_1 i_2 \cdots i_k, j_1 j_2 \cdots j_k} M_{i_1 i_2 \cdots i_k, j_1 j_2 \cdots j_k},$$

where the symbol $\sum_{(i)}$ indicates that the summation is taken over all possible permutations in the following way. The first set of subscripts $i_1 i_2 \ldots i_k$ is from n indices $12 \ldots n$ taken k at a time with the restriction $i_1 < i_2 \cdots < i_k$. The second set subscripts $j_1 j_2 \ldots j_k$ are chosen arbitrarily but remain fixed for each term of the expansion. This formula is general, but is seldom needed for the evaluation of a determinant.

Example 4.6.1. Evaluate

$$D_4 = \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 2 & 5 \\ 2 & 1 & 1 & 3 \\ 1 & 3 & 0 & 2 \end{vmatrix}$$

by (a) expansion with minors formed from the first two columns, (b) expansion with minors formed from the second and fourth rows.

Solution 4.6.1. (a)

$$D_4 = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix}$$

$$= -2 - 0 + 5 + 6 - 24 + 65 = 50.$$

(b)

$$\begin{aligned}D_4 &= (-1)^{2+4+1+2}\begin{vmatrix}1&0\\1&3\end{vmatrix}\cdot\begin{vmatrix}3&1\\1&3\end{vmatrix}+(-1)^{2+4+1+3}\begin{vmatrix}1&2\\1&0\end{vmatrix}\cdot\begin{vmatrix}1&1\\1&3\end{vmatrix}\\ &\quad+(-1)^{2+4+1+4}\begin{vmatrix}1&5\\1&2\end{vmatrix}\cdot\begin{vmatrix}1&3\\1&1\end{vmatrix}+(-1)^{2+4+2+3}\begin{vmatrix}0&2\\3&0\end{vmatrix}\cdot\begin{vmatrix}2&1\\2&3\end{vmatrix}\\ &\quad+(-1)^{2+4+2+4}\begin{vmatrix}0&5\\3&2\end{vmatrix}\cdot\begin{vmatrix}2&3\\2&1\end{vmatrix}+(-1)^{2+4+3+4}\begin{vmatrix}2&5\\0&2\end{vmatrix}\cdot\begin{vmatrix}2&1\\2&1\end{vmatrix}\\ &= -24-4-6+24+60-0=50.\end{aligned}$$

4.7 Multiplication of Determinants of the Same Order

If $|A|$ and $|B|$ are determinants of order n, then the product

$$|A|\cdot|B| = |C|$$

is a determinant of the same order. Its elements are given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

(As we shall show in Chap. 5, this is the rule of multiplying two matrices.)

For second-order determinants, this relation is expressed as

$$|A|\cdot|B| = \begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}\cdot\begin{vmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{vmatrix} = \begin{vmatrix}(a_{11}b_{11}+a_{12}b_{21})&(a_{11}b_{12}+a_{12}b_{22})\\(a_{21}b_{11}+a_{22}b_{21})&(a_{21}b_{12}+a_{22}b_{22})\end{vmatrix}.$$

To prove this, we use the property of block diagonal determinants.

$$|A|\cdot|B| = \begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}\cdot\begin{vmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{vmatrix} = \begin{vmatrix}a_{11}&a_{12}&0&0\\a_{21}&a_{22}&0&0\\-1&0&b_{11}&b_{12}\\0&-1&b_{21}&b_{22}\end{vmatrix}.$$

Multiplying the elements in the first column by b_{11} and the elements in the second column by b_{21} and then add them to the corresponding elements in the third column, we obtain

$$|A|\cdot|B| = \begin{vmatrix}a_{11}&a_{12}&(a_{11}b_{11}+a_{12}b_{21})&0\\a_{21}&a_{22}&(a_{21}b_{11}+a_{22}b_{21})&0\\-1&0&0&b_{12}\\0&-1&0&b_{22}\end{vmatrix}.$$

In the same way, we multiply the elements in the 1st column by b_{12} and the elements in the second column by b_{22} and then add them to the corresponding elements in the fourth column, it become

$$|A| \cdot |B| = \begin{vmatrix} a_{11} & a_{12} & (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ a_{21} & a_{22} & (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}.$$

By (4.48)

$$|A| \cdot |B| = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{vmatrix}$$
$$= \begin{vmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{vmatrix},$$

which is the desired result. This procedure is applicable to determinants of any order. (This property is of considerable importance, we will revisit this problem for determinant of higher order in the chapter on matrices.)

Example 4.7.1. Show that

$$\begin{vmatrix} b^2 + c^2 & ab & ca \\ ab & a^2 + b^2 & bc \\ ca & bc & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2.$$

Solution 4.7.1.

$$\begin{vmatrix} b^2 + c^2 & ab & ca \\ ab & a^2 + b^2 & bc \\ ca & bc & a^2 + b^2 \end{vmatrix} = \begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix} \cdot \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = (-2abc)^2 = 4a^2b^2c^2.$$

4.8 Differentiation of Determinants

Occasionally, we require an expression for the derivative of a determinant. If the derivative is with respect to a particular element a_{ij}, then

$$\frac{\partial D_n}{\partial a_{ij}} = C_{ij},$$

where C_{ij} is the cofactor of a_{ij}, since

$$D_n = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for} \ \ 1 \le i \le n.$$

Suppose the elements are functions of a parameter s, the derivative of D_n with respect to s is then given by

$$\frac{\mathrm{d}D_n}{\mathrm{d}s} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial D_n}{\partial a_{ij}} \frac{\mathrm{d}a_{ij}}{\mathrm{d}s} = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} \frac{da_{ij}}{\mathrm{d}s}.$$

For example

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{j=1}^{3} a_{1j}C_{1j} = \sum_{j=1}^{3} a_{2j}C_{2j} = \sum_{j=1}^{3} a_{3j}C_{3j},$$

$$\begin{aligned} \frac{\mathrm{d}D_3}{\mathrm{d}s} &= \sum_{j=1}^{3} \frac{\mathrm{d}a_{1j}}{\mathrm{d}s} C_{1j} + \sum_{j=1}^{3} \frac{\mathrm{d}a_{2j}}{\mathrm{d}s} C_{2j} + \sum_{j=1}^{3} \frac{\mathrm{d}a_{3j}}{\mathrm{d}s} C_{3j} \\ &= \begin{vmatrix} \frac{\mathrm{d}a_{11}}{\mathrm{d}s} & \frac{\mathrm{d}a_{12}}{\mathrm{d}s} & \frac{\mathrm{d}a_{13}}{\mathrm{d}s} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \frac{\mathrm{d}a_{21}}{\mathrm{d}s} & \frac{\mathrm{d}a_{22}}{\mathrm{d}s} & \frac{\mathrm{d}a_{23}}{\mathrm{d}s} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \frac{\mathrm{d}a_{31}}{\mathrm{d}s} & \frac{\mathrm{d}a_{32}}{\mathrm{d}s} & \frac{\mathrm{d}a_{33}}{\mathrm{d}s} \end{vmatrix}. \end{aligned}$$

Example 4.8.1. If $D_2 = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$, find $\frac{\mathrm{d}D_2}{\mathrm{d}x}$.

Solution 4.8.1.

$$\frac{\mathrm{d}D_2}{\mathrm{d}x} = \begin{vmatrix} -\sin x & \cos x \\ -\sin x & \cos x \end{vmatrix} + \begin{vmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{vmatrix} = 0.$$

This is an obvious result, since $D_2 = \cos^2 x + \sin^2 x = 1$.

4.9 Determinants in Geometry

It is well known in analytic geometry that a straight line in the xy-plane is represented by the equation

$$ax + by + c = 0. \tag{4.56}$$

The line is uniquely defined by two points. If the line goes through two points (x_1, y_1) and (x_2, y_2), then both of them have to satisfy the equation

$$ax_1 + by_1 + c = 0, \tag{4.57}$$
$$ax_2 + by_2 + c = 0. \tag{4.58}$$

These (4.56)–(4.58) may be regarded as a system in the unknowns a, b, c which cannot all vanish if (4.56) represents a line. Hence the coefficient determinant must vanish:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{4.59}$$

It can be easily shown that (4.59) is indeed the familiar equation of a line. Expanding (4.59) by the third column, we have

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = (x_1y_2 - x_2y_1) - (xy_2 - x_2y) + (xy_1 - x_1y) = 0.$$

This equation can be readily transformed into (4.56) with $a = y_1 - y_2,\ b = x_2 - x_1,\ c = x_1y_2 - x_2y_1$. Or it can be put in form

$$y = mx + y_0,$$

where $m = \frac{y_2-y_1}{x_2-x_1}$ is the slope and $y_0 = y_1 - mx_1$ is the y-axis intercept.

It follows from (4.59) that a necessary and sufficient condition for three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) to lie on a line is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{4.60}$$

Now if the three points are not on a line, then they form a triangle and the determinant (4.60) is not equal to zero. In that case it is interesting to ask what does the determinant represent. Since it has the dimension of an area, this strongly suggests that the determinant is related to the area of the triangle.

The area of the triangle formed by three points $A(x_1, y_1), B(x_2, y_2)$, and $C(x_3, y_3)$ shown in Fig. 4.6 is seen to be

$$\text{Area } ABC = \text{Area } AA'C'C + \text{Area } CC'B'B - \text{Area } AA'B'B.$$

The area of a trapezoid is equal to half of the product of its altitude and the sum of the parallel sides:

$$\text{Area } AA'C'C = \frac{1}{2}(x_3 - x_1)(y_1 + y_3),$$

$$\text{Area } CC'B'B = \frac{1}{2}(x_2 - x_3)(y_2 + y_3),$$

$$\text{Area } AA'B'B = \frac{1}{2}(x_2 - x_1)(y_1 + y_2).$$

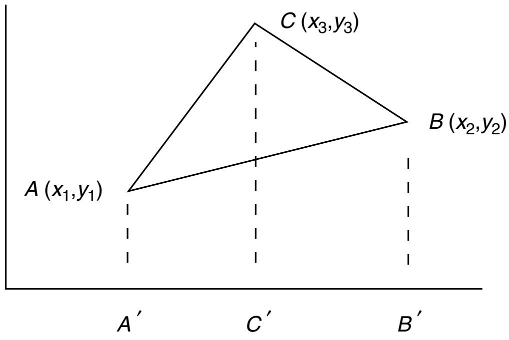

Fig. 4.6. The area of ABC is equal to the sum of the trapezoids $AA'C'C$ and $CC'B'B$ minus the trapezoid $AA'B'B$. As a consequence, the area ABC can be represented by a determinant

Hence

$$\begin{aligned}\text{Area } ABC &= \frac{1}{2}[(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) \\ &\quad -(x_2 - x_1)(y_1 + y_2)] \\ &= \frac{1}{2}\left[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right] \\ &= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \qquad (4.61)\end{aligned}$$

Notice the order of the points ABC in the figure is counterclockwise. If it is clockwise, the positions of B and C are interchanged. This will result in the interchange of row 2 and row 3 in the determinant. As a consequence, a minus sign will be introduced. Thus we conclude that if the three vertices of a triangle are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2 \times \text{Area of } ABC, \qquad (4.62)$$

where the $+$ or $-$ sign is chosen according to the vertices being numbered consecutively in the counterclockwise or the clockwise direction.

Example 4.9.1. Use a determinant to find the circle that passes through (2,6), (6,4), (7,1).

Solution 4.9.1. The general expression of a circle is

$$a(x^2 + y^2) + bx + cy + d = 0.$$

The three points must all satisfy this equation

$$a(x_1^2+y_1^2)+bx_1+cy_1+d=0,$$

$$a(x_2^2+y_2^2)+bx_2+cy_2+d=0,$$

$$a(x_3^2+y_3^2)+bx_3+cy_3+d=0.$$

These equations may be regarded as a system of equations in the unknowns a, b, c, d which cannot all be zero. Hence the coefficient determinant must vanish

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Put in the specific values

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 40 & 2 & 6 & 1 \\ 52 & 6 & 4 & 1 \\ 50 & 7 & 1 & 1 \end{vmatrix} = 0.$$

Replacing the first row by (row 1 − row 2), and the third row by (row 3 − row 2) and the fourth row by (row 4 − row 2), we have

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 40 & 2 & 6 & 1 \\ 52 & 6 & 4 & 1 \\ 50 & 7 & 1 & 1 \end{vmatrix} = \begin{vmatrix} (x^2+y^2-40) & (x-2) & (y-6) & 0 \\ 40 & 2 & 6 & 1 \\ 12 & 4 & -2 & 0 \\ 10 & 5 & -5 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} (x^2+y^2-40) & (x-2) & (y-6) \\ 12 & 4 & -2 \\ 10 & 5 & -5 \end{vmatrix}$$

$$= -10(x^2+y^2-40)+40\,(x-2)+20\,(y-6)=0,$$

or

$$x^2+y^2-40-4(x-2)-2(y-6)=0.$$

which can be written as

$$(x-2)^2+(y-1)^2=25.$$

So the circle is centered at $x=2,\ y=1$ with a radius of 5.

Example 4.9.2. What is the area of the triangle whose vertices are $(-2,1)$, $(4,3), (0,0)$?

Solution 4.9.2.

$$\text{Area} = \begin{vmatrix} -2 & 1 & 1 \\ 4 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -10.$$

The area of the triangle is 10 and the order of the vertices is clockwise.

Exercises

1. Use determinants to solve for x, y, z from the following system of equations:

$$\begin{aligned} 3x + 6z &= 51, \\ 12y - 6z &= -6, \\ x - y - z &= 0. \end{aligned}$$

   Ans. $x = 7,\ y = 2,\ z = 5.$

2. By applying the Kirchhoff's rule to a electric circuit, the following equations are obtained for the currents i_1, i_2, i_3 in three branches

$$\begin{aligned} i_1 R_1 + i_3 R_3 &= V_A \\ i_2 R_2 + i_3 R_3 &= V_C \\ i_1 + i_2 - i_3 &= 0. \end{aligned}$$

   Express i_1, i_2, i_3 in terms of resistance R_1, R_2, R_3, and voltage source V_A, V_C.
   Ans.

$$\begin{aligned} i_1 &= \frac{(R_2 + R_3)V_A - R_3 V_C}{R_1 R_2 + R_1 R_3 + R_2 R_3}, \\ i_2 &= \frac{(R_1 + R_3)V_C - R_3 V_A}{R_1 R_2 + R_1 R_3 + R_2 R_3}, \\ i_3 &= \frac{R_2 V_A + R_1 V_C}{R_1 R_2 + R_1 R_3 + R_2 R_3}. \end{aligned}$$

3. Find the value of the following fourth-order determinant (which happens to be formed from one of the matrices appearing in Dirac's relativistic electron theory)

$$D_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}.$$

Ans. 1.

4. Without computation, show that a skew-symmetric determinant of odd order is zero

$$D_{\mathrm{ss}} = \begin{vmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ -b & -e & 0 & h & i \\ -c & -f & -h & 0 & j \\ -d & -g & -i & -j & 0 \end{vmatrix} = 0.$$

[Hint: $D^{\mathrm{T}} = D$ and $(-1)^n D_{\mathrm{ss}} = D_{\mathrm{ss}}^{\mathrm{T}}$.

5. Show that $\begin{vmatrix} a & d & 2a-3d \\ b & e & 2b-3e \\ c & f & 2c-3f \end{vmatrix} = 0.$

6. Determine x such that $\begin{vmatrix} 1 & 2 & -3 \\ -x & 1+3x & 3-x \\ 0 & -6 & 5 \end{vmatrix} = 36.$

Ans. 13.

7. The development of the determinant D_n on the ith row elements a_{ik} is $\sum_{k=1}^{n} a_{ik}C_{ik}$, where C_{ik} is the cofactor of a_{ik}. Show that

$$\sum_{k=1}^{n} a_{jk}C_{ik} = 0 \quad \text{for } j \neq i.$$

[Hint: The expansion is another determinant with two identical rows.]

8. Evaluate the following determinant by a development on (a) the first column, (b) the second row

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}.$$

Ans. 1.

9. Use the properties of determinants to transform the determinant in problem 6 into a triangular form and then evaluate it as the product of the diagonal elements.

10. Evaluate the determinant in problem 6 by expanding it in terms of the 2×2 minors formed from the first two columns.

11. Evaluate the determinant

$$D_5 = \begin{vmatrix} 3 & -1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 3 & -6 & 1 \end{vmatrix}.$$

Ans. 3080.
[The quickest way to evaluate is to expand it in terms of the 2×2 minors formed from the first two columns.]

12. Without expanding, show that

$$\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

[Hint: Add row 1 and row 2, factor out $(x+y+z)$.]

13. Show that (a)

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (xy+yz+zx)\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

[Hint: Replace row 3 successively by $x\cdot$ row 1 + row 3, then by $y\cdot$ row 1 + row 3, then by $z\cdot$ row 1 + row 3. Express the result as a sum of two determinants, one of them is equal to zero.]
(b) Use the result of the Vandermonde determinant to show that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (xy+yz+zx)(x-y)(y-z)(z-x).$$

14. State the reason for each step of the following identity:

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = \begin{vmatrix} 2a & -2b & -a & b \\ 2b & 2a & -b & -a \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{vmatrix}$$

$$= \begin{vmatrix} 2a & -2b \\ 2b & 2a \end{vmatrix} \cdot \begin{vmatrix} c & -d \\ d & c \end{vmatrix} = 4(a^2+b^2)(c^2+d^2)$$

15. State the reason for each step of the following identity

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = \begin{vmatrix} (a+b) & b & (c+d) & d \\ (b+a) & a & (d+c) & c \\ (c+d) & d & (a+b) & b \\ (d+c) & c & (b+a) & a \end{vmatrix} = \begin{vmatrix} (a+b) & b & (c+d) & d \\ 0 & a-b & 0 & c-d \\ (c+d) & d & (a+b) & b \\ 0 & c-d & 0 & a-b \end{vmatrix}$$

$$= \begin{vmatrix} (a+b) & (c+d) & b & d \\ (c+d) & (a+b) & d & b \\ 0 & 0 & a-b & c-d \\ 0 & 0 & c-d & a-b \end{vmatrix}$$

$$= [(a+b)^2 - (c+d)^2][(a-b)^2 - (c-d)^2].$$

16. Show and state the reason for each step of the following identity

$$\begin{vmatrix} 0 & 1 & 2 & 3 & \cdots & n-1 \\ 1 & 0 & 1 & 2 & \cdots & n-2 \\ 2 & 1 & 0 & 1 & \cdots & n-3 \\ 3 & 2 & 1 & 0 & \cdots & n-4 \\ . & . & . & . & \cdots & . \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{vmatrix} = -(-2)^{n-2}(n-1).$$

[Hint: 1. Replace column 1 by column 1 + last column. 2. Factor out $(n-1)$. 3. Replace row i by row i − row$(i-1)$, starting with the last row. 3. Replace row i by row i + row 2. 4. Evaluating the triangular determinant.]

17. Evaluate the following determinant

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ . & . & . & \cdots & . \\ (n-1)n+1 & (n-1)n+2 & (n-1)n+3 & \cdots & n^2 \end{vmatrix}.$$

Ans. For $n = 1,\ D_1 = 1;\ n = 2,\ D_2 = -2;\ n \geq 3,\ D_n = 0.$
[Hint: For $n \geq 3$, replace row i by row i − row$(i-1)$.]

18. Use the rule of product of two determinants of same order to show the

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & a^2+b^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} b^2+ac & bc & c^2 \\ ab & 2ac & bc \\ a^2 & ab & b^2+ac \end{vmatrix}.$$

[Hint:

$$\begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix} \cdot \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = \begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix} \cdot \begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix}.$$

19. If $f(s)$ is given by the following determinants, without the expansion of $f(x)$ find $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$

$$\text{(a)}\quad f(x) = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{-x} & 1 \\ \mathrm{e}^x & -\mathrm{e}^{-x} & 0 \\ \mathrm{e}^x & -\mathrm{e}^{-x} & x \end{vmatrix}; \qquad \text{(b)}\quad f(x) = \begin{vmatrix} \cos x & \sin x & \ln|x| \\ -\sin x & \cos x & \frac{1}{x} \\ -\cos x & -\sin x & -\frac{1}{x^2} \end{vmatrix}.$$

Ans. (a) -2; (b) $1/x + 2/x^3$.

20. The vertices of a triangle are $(0,t), (3t,0), (t,2t)$. Find a formula for the area of the triangle.

Ans. $2t$.

21. The equation representing a plane is given by $ax+by+cz+d=0$. Find the plane that goes through $(1,1,1), (5,0,5), (3,2,6)$.

Ans. $3x+4y-2z-5=0$.

5

Matrix Algebra

Matrices were introduced by British mathematician Arthur Cayley (1821–1895). The method of matrix algebra has extended far beyond mathematics into almost all disciplines of learning. In physical sciences, matrix is not only useful, but also essential in handling many complicated problems. These problems are mainly in three categories. First in the theory of transformation, second in the solution of systems of linear equations, and third in the solution of eigenvalue problems. In this chapter, we shall discuss various matrix operations and different situations in which they can be applied.

5.1 Matrix Notation

In this section, we shall define a matrix and discuss some of the simple operations by which two or more matrices can be combined.

5.1.1 Definition

Matrices

A rectangular array of elements is called a *matrix*. The array is usually enclosed within curved or square brackets. Thus, the rectangular arrays

$$\begin{pmatrix} 4 & 7 \\ 12 & 6 \\ -9 & 3 \end{pmatrix}, \quad \begin{pmatrix} x+\mathrm{i}y \\ x-\mathrm{i}y \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

are examples of a matrix. It is convenient to think of every element of a matrix as belonging to a certain row and a certain column of the matrix. If a matrix has m rows and n columns, the matrix is said of order m by n, or $m \times n$. Every element of a matrix can be uniquely characterized by a row index and a column index. It is convenient to write a $m \times n$ matrix as

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where a_{ij} is the element of ith row and jth column, it may be real or complex number or functions. The elements may even be matrices themselves, in which case the elements are called *submatrices* and the whole matrix is said to be partitioned.

Thus, if the first matrix in (5.1) is called matrix A, then A is a 3×2 matrix, it has three rows: $(4\ 7), (12\ 6), (-9\ 3)$ and two columns: $\begin{pmatrix} 4 \\ 12 \\ -9 \end{pmatrix}, \begin{pmatrix} 7 \\ 6 \\ 3 \end{pmatrix}$. Its elements are $a_{11} = 4, a_{12} = 7, a_{21} = 12, a_{22} = 6, a_{31} = -9$, and $a_{32} = 3$.

Some times it is convenient to use the notation

$$A = (a_{ij})_{m \times n}$$

to indicate that A is a $m \times n$ matrix. The elements a_{ij} can also be expressed as

$$a_{ij} = (A)_{ij}\,.$$

5.1.2 Some Special Matrices

There are some special matrices, which are named after their appearances.

Zero Matrix

A matrix of arbitrary order is said to be a zero matrix if and only if every element of the matrix equals zero. A zero matrix is sometimes called a *null matrix.*

Row Matrix

A row matrix has only one row, such as $(1\ 0\ 3)$. A row matrix is also called a *row vector.* If it is called row vector, the elements of the matrix are usually referred as components.

Column Matrix

A column matrix has only one column, such as $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$. A column matrix is also called a *column vector.* Again if it is called column vector, the elements of the matrix are usually called the components of the vector.

Square Matrix

A matrix is said to be a square matrix if the number of rows equals the number of columns. A square matrix of order n simply means it has n rows and n columns. Square matrix is of particular importance. We will be dealing mostly with square matrices together with column and row matrices.

For a square matrix A, we can calculate the determinant

$$\det(A) = |A|,$$

as defined in Chap. 4. Matrix is not a determinant. Matrix is an array of numbers, determinant is a single number. The determinant of a matrix can only be defined for a square matrix.

Let $A = (a_{ij})_n$ be a square matrix of order n. The diagonal going from the top left corner to the bottom right corner of the matrix, its elements $a_{11}, a_{22}, \ldots, a_{nn}$, are called the *diagonal elements*. All the remaining elements a_{ij} for $i \neq j$ are called the *off-diagonal elements*.

There are several special square matrices that are of interest.

Diagonal Matrix

A diagonal matrix is a square matrix whose diagonal elements are not all equal to zero, but off-diagonal elements are all zero. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

are diagonal matrices. Therefore for a diagonal matrix

$$(A)_{ij} = a_{ii}\delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

This kind of notation may seem to be redundant, as a diagonal matrix can easily be visualized. However, this notation is useful in manipulating matrices as we shall see later.

Constant Matrix

If all elements of a diagonal matrix happen to be equal to each other, it is said to be a constant matrix or a *scalar matrix*.

Unit Matrix

If the elements of a constant matrix are equal to unity, then it is a unit matrix. A unit matrix is also called the *Identity matrix,* denoted by I, that is

$$I=\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Triangular Matrix

A square matrix having only zero elements on one side of the principal diagonal is a triangular matrix. Thus

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 3 & 0 \end{pmatrix}$$

are examples of a triangular matrix. A matrix for which $a_{ij} = 0$ for $i > j$ is called a *right-triangular matrix* or a *upper triangular matrix*, such as matrix A above. Whereas a matrix with $a_{ij} = 0$ for $i < j$ is called a *left-triangular matrix* or a *lower triangular matrix*, such as matrix B. If all the principal diagonal elements are zero, the matrix is a *strictly triangular matrix*, such as matrix C. Diagonal matrix, identity matrix as well as zero matrix are all triangular matrices.

5.1.3 Matrix Equation

Equality

Two matrices A and B are equal to each other if and only if, every elements of A is equal to the corresponding element of B. Clearly A and B must be of the same order, in other words they must have the same rows and columns. Thus if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix},$$

we see that

$$A \neq B, \quad B \neq C, \quad C \neq A.$$

Therefore, a matrix equation $A = B$ means that A and B are of the same order and their corresponding elements are equal, i.e., $a_{ij} = b_{ij}$. For example, the equation

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 3t & 1+2t \\ 4t^2 & 0 \end{pmatrix}$$

means $x_1 = 3t, y_1 = 4t^2, x_2 = 1 + 2t, y_2 = 0$.

With this understanding, often we can use a single matrix equation to replace a set of equations. This will not only simplify the writing but will also enable us to systematically manipulate these equations.

Addition and Subtraction

We may now define the addition and subtraction of two matrices of the same order. The sum of two matrices A and B is another matrix C. By definition

$$A + B = C$$

means

$$c_{ij} = a_{ij} + b_{ij}.$$

For example,

$$A = \begin{pmatrix} 1 & 3 & 12 \\ -2 & 4 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} -10 & 5 & -6 \\ 7 & 3 & 2 \end{pmatrix},$$

then

$$\begin{aligned} A + B &= \begin{pmatrix} (1-10) & (3+5) & (12-6) \\ (-2+7) & (4+3) & (-6+2) \end{pmatrix} = \begin{pmatrix} -9 & 8 & 6 \\ 5 & 7 & -4 \end{pmatrix}, \\ A - B &= \begin{pmatrix} (1+10) & (3-5) & (12+6) \\ (-2-7) & (4-3) & (-6-2) \end{pmatrix} = \begin{pmatrix} 11 & -2 & 18 \\ -9 & 1 & -8 \end{pmatrix}. \end{aligned}$$

The sum of several matrices is obtained by repeated addition. Since matrix addition is merely the addition of corresponding elements, it does not matter in which order we add several matrices. To be explicit, if A, B, C are three $m \times n$ matrices, then both commutative and associative laws hold

$$\begin{aligned} A + B &= B + A, \\ A + (B + C) &= (A + B) + C. \end{aligned}$$

Multiplication by a Scalar

It is possible to combine a matrix of arbitrary order and a scalar by scalar multiplication. If A is a matrix of order $m \times n$

$$A = (a_{ij})_{m \times n}$$

and c a scalar, we define cA to be another $m \times n$ matrix such that

$$cA = (ca_{ij})_{m \times n}.$$

For example, if

$$A = \begin{pmatrix} 1 & -3 & 5 \\ -2 & 4 & -6 \end{pmatrix},$$

then

$$-2A = \begin{pmatrix} -2 & 6 & -10 \\ 4 & -8 & 12 \end{pmatrix}.$$

The scalar can be a real number, a complex number, or a function, but it cannot be a matrix quantity.

Note the difference between the scalar multiplication of a square matrix cA and the scalar multiplication of its determinant $c|A|$. For cA, c is multiplied to every elements of A, whereas for $c|A|$, c is only multiplied to the elements of a single column or a single row. Thus, if A is a square matrix of order n, then

$$\det(cA) = c^n |A| .$$

5.1.4 Transpose of a Matrix

If the rows and columns are interchanged, the resulting matrix is called the *transposed matrix*. The transposed matrix is denoted by $\widetilde{A}$, called A tilde, or by A^{T}. Usually, but not always, the transpose of a single matrix is denoted by the tilde and the transpose of the product of a number of matrices by the superscript T.

Thus, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

then

$$\widetilde{A} = A^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

By definition, if we transpose the matrix twice, we should get the original matrix, i.e.,

$$\widetilde{A}^{\mathrm{T}} = A.$$

Using index notation, this means

$$\left(\widetilde{A}\right)_{ij} = (A)_{ji}\,, \quad \left(\widetilde{A}^{\mathrm{T}}\right)_{ij} = (A)_{ij}\,.$$

It is clear that the transpose of $m \times n$ matrix is a $n \times m$ matrix. The transpose of a square matrix is another square matrix. The transpose of a column matrix is a row matrix, and the transpose of a row matrix is a column matrix.

Symmetric Matrix

A symmetric matrix is a matrix that is equal to its transpose, i.e.,

$$A = \widetilde{A},$$

which means

$$a_{ij} = a_{ji}.$$

It is symmetric with respect to its diagonal. A symmetric matrix must be a square matrix.

Antisymmetric Matrix

An antisymmetric matrix is a matrix that is equal to the negative of its transpose, i.e.,

$$A = -\widetilde{A},$$

which means

$$a_{ij} = -a_{ji}.$$

Thus the diagonal elements of an antisymmetric matrix must all be zero. An antisymmetric matrix must also be a square matrix. Antisymmetric is also known as *skew-symmetric*.

Decomposition of a Square Matrix

Any square matrix can be written as the sum of a symmetric and an antisymmetric matrix. Clearly

$$A = \frac{1}{2}\left(A + \widetilde{A}\right) + \frac{1}{2}\left(A - \widetilde{A}\right)$$

is an identity. Furthermore, let

$$A_{\mathrm{s}} = \frac{1}{2}\left(A + \widetilde{A}\right), \quad A_{\mathrm{a}} = \frac{1}{2}\left(A - \widetilde{A}\right),$$

then A_{s} is symmetric, since

$$A_{\mathrm{s}}^{\mathrm{T}} = \frac{1}{2}\left(A^{\mathrm{T}} + \widetilde{A}^{\mathrm{T}}\right) = \frac{1}{2}\left(\widetilde{A} + A\right) = A_{\mathrm{s}},$$

and A_{a} is antisymmetric, since

$$A_{\mathrm{a}}^{\mathrm{T}} = \frac{1}{2}\left(A^{\mathrm{T}} - \widetilde{A}^{\mathrm{T}}\right) = \frac{1}{2}\left(\widetilde{A} - A\right) = -A_{\mathrm{a}}.$$

Therefore

$$A = A_{\mathrm{s}} + A_{\mathrm{a}},$$
$$\widetilde{A} = A_{\mathrm{s}} - A_{\mathrm{a}}.$$

Example 5.1.1. Express the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix}$$

as the sum of a symmetric matrix and an antisymmetric matrix.

Solution 5.1.1.

$$A = A_\mathrm{s} + A_\mathrm{a},$$

$$A_\mathrm{s} = \frac{1}{2}\left[\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}\right] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 2 \end{pmatrix},$$

$$A_\mathrm{a} = \frac{1}{2}\left[\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

5.2 Matrix Multiplication

5.2.1 Product of Two Matrices

The multiplication, or product, of two matrices is not a simple extension of the concept of multiplication of two numbers. The definition of matrix multiplication is motivated by the theory of linear transformation, which we will briefly discuss in Sect. 5.3.

Two matrices A and B can be multiplied together only if the number of columns of A is equal the number of rows of B. The matrix multiplication depends on the order in which the matrices occur in the product. For example, if A is of order $l \times m$, and B is of order $m \times n$, then the product matrix AB is defined but the product BA, in that order is not unless $m = l$. The multiplication is defined as follows. If

$$A = (a_{ij})_{l\times m}, \quad B = (b_{ij})_{m\times n},$$

then $AB = C$ means that C is a matrix of order $l \times n$ and

$$C = (c_{ij})_{l\times n},$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}.$$

So the element of the C matrix at ith row and jth column is the sum of all the products of the elements of ith row of A and the corresponding elements of jth column of B. Thus, if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}, \quad C = AB,$$

then

$$C = \begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) & (a_{11}b_{13} + a_{12}b_{23}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) & (a_{21}b_{13} + a_{22}b_{23}) \end{pmatrix}.$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
a_{i1} & a_{i2} & \cdots & a_{im} \\
\cdot & \cdot & & \cdot \\
a_{l1} & a_{l2} & \cdots & a_{lm}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} & \cdot\cdot & b_{1j} & \cdot\cdot & b_{1n} \\
b_{21} & b_{22} & \cdot\cdot & b_{2j} & \cdot\cdot & b_{2n} \\
\cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \\
b_{m1} & b_{m2} & \cdot\cdot & b_{mj} & \cdot\cdot & b_{mn}
\end{pmatrix}
=
\begin{pmatrix}
c_{11} & c_{12} & \cdot\cdot & c_{1j} & \cdot\cdot & c_{1n} \\
c_{21} & c_{22} & \cdot\cdot & c_{2j} & \cdot\cdot & c_{2n} \\
\cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \\
c_{i1} & c_{i2} & \cdot\cdot & \boxed{c_{ij}} & \cdot\cdot & c_{in} \\
\cdot & \cdot & & \cdot & & \cdot \\
c_{l1} & c_{l2} & \cdot\cdot & c_{lj} & \cdot\cdot & c_{ln}
\end{pmatrix}
$$

Fig. 5.1. Illustration of matrix multiplication. The number of columns of A must equal the number of rows of B for the multiplication $AB = C$ to be defined. The element at ith row and jth column of C is given by $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$

The multiplication of two matrices is illustrated in Fig. 5.1.

If the product AB is defined, A and B are said to be *comformable* (or *compatible*). If the matrix product AB is defined, the product BA is not necessarily defined. Given two matrices A and B, both the products of AB and BA will be possible if, for example, A is of order $m \times n$ and B is of order $n \times m$. AB will be of order $m \times m$, and BA of order $n \times n$. Clearly if $m \neq n$, AB cannot equal to BA, since they are of different order. Even if $n = m$, AB is still not necessarily equal to BA. The following examples will make this clear.

Example 5.2.1. Find the product AB, if

$$
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix}.
$$

Solution 5.2.1.

$$
\begin{aligned}
AB &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \\
&= \begin{pmatrix} (1 \times 3 + 2 \times 4) & (1 \times 2 + 2 \times 5) & (1 \times 1 + 2 \times 6) \\ (3 \times 3 + 4 \times 4) & (3 \times 2 + 4 \times 5) & (3 \times 1 + 4 \times 6) \end{pmatrix} \\
&= \begin{pmatrix} 11 & 12 & 13 \\ 25 & 26 & 27 \end{pmatrix}.
\end{aligned}
$$

Here A is 2×2 and B is 2×3, so that AB comes out 2×3, whereas BA is not defined.

Example 5.2.2. Find the product AB, if

$$
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.
$$

Solution 5.2.2.

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 + 12 \\ 15 + 24 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}.$$

Here AB is a column matrix and BA is not defined.

Example 5.2.3. Find AB and BA, if

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

Solution 5.2.3.

$$AB = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 + 6 + 12 \end{pmatrix} = (20),$$

$$BA = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}.$$

This example dramatically shows that $AB \neq BA$.

Example 5.2.4. Find AB and BA, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

Solution 5.2.4.

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 13 & 16 \\ 29 & 36 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 15 & 22 \\ 23 & 34 \end{pmatrix}.$$

Clearly

$$AB \neq BA.$$

Example 5.2.5. Find AB and BA, if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solution 5.2.5.

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$BA = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Not only $AB \neq BA$, but also $AB = 0$ does not necessarily imply $A = 0$ or $B = 0$ or $BA = 0$.

Example 5.2.6. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}$$

show that

$$AB = AC.$$

Solution 5.2.6.

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix},$$

$$AC = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}.$$

This example shows that $AB = AC$ can hold without $B = C$ or $A = 0$.

5.2.2 Motivation of Matrix Multiplication

Much of the usefulness of matrix algebra is due to its multiplication property. The definition of matrix multiplication, as we have seen, seems to be "unnatural" and somewhat complicated. The motivation of this definition comes from the "linear transformations." It provides a simple mechanism for changing variables. For example, suppose

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \tag{5.2a}$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \tag{5.2b}$$

and further

$$z_1 = b_{11}y_1 + b_{12}y_2, \tag{5.3a}$$
$$z_2 = b_{21}y_1 + b_{22}y_2. \tag{5.3b}$$

In these equations the x's and the y's are variables, while the a's and the b's are constants. The x's are related to the y's by the first set of equations, and the y's are related to the z's by the second set of equations. To find out how the x's are related to the z's, we must substitute the values of the y's given by the first set of equations into the second set of equations

$$z_1 = b_{11}\left(a_{11}x_1 + a_{12}x_2 + a_{13}x_3\right) + b_{12}\left(a_{21}x_1 + a_{22}x_2 + a_{23}x_3\right), \quad (5.4a)$$
$$z_2 = b_{21}\left(a_{11}x_1 + a_{12}x_2 + a_{13}x_3\right) + b_{22}\left(a_{21}x_1 + a_{22}x_2 + a_{23}x_3\right). \quad (5.4b)$$

By multiplying them out and collecting coefficients, they become

$$\begin{aligned} z_1 &= (b_{11}a_{11} + b_{12}a_{21})x_1 \\ &\quad +(b_{11}a_{12} + b_{12}a_{22})x_2 + (b_{11}a_{13} + b_{12}a_{23})x_3, \end{aligned} \quad (5.5a)$$
$$\begin{aligned} z_2 &= (b_{21}a_{11} + b_{22}a_{21})x_1 \\ &\quad +(b_{21}a_{12} + b_{22}a_{22})x_2 + (b_{21}a_{13} + b_{22}a_{23})x_3. \end{aligned} \quad (5.5b)$$

Now, using matrix notation, (5.2) can be written simply as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad (5.6)$$

and (5.3) as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad (5.7)$$

Not only the coefficients of x_1, x_2, and x_3 in (5.5) are precisely the elements of the matrix product

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

but also they are located in the proper position. In other words, (5.5) can be obtained by simply substituting $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ from (5.6) into (5.7)

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (5.8)$$

What we have shown here is essentially two things. First, matrix multiplication is defined in such a way that linear transformation can be written in compact forms. Second, if we substitute linear transformations into each other, we can obtain the composite transformation simply by multiplying coefficient matrices in the right order. This kind of transformation is not only common in mathematics, but is also extremely important in physics. We will discuss some of them in later sections.

5.2.3 Properties of Product Matrices

Transpose of a Product Matrix

A result of considerable importance in matrix algebra is that the transpose of the product of two matrices equals the product of the transposed matrices taken in reverse order,

$$(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}. \tag{5.9}$$

To prove this we must show that every element of the left-hand side is equal to the corresponding element in the right-hand side. The ijth element of the left-hand side of (5.9) is given by

$$\left((AB)^{\mathrm{T}}\right)_{ij} = (AB)_{ji} = \sum_k (A)_{jk}(B)_{ki}. \tag{5.10}$$

The ijth element of the left-hand side of (5.9) is

$$\begin{aligned}\left(\widetilde{B}\widetilde{A}\right)_{ij} &= \sum_k \left(\widetilde{B}\right)_{ik}\left(\widetilde{A}\right)_{kj} = \sum_k (B)_{ki}(A)_{jk} \\ &= \sum_k (A)_{jk}(B)_{ki},\end{aligned} \tag{5.11}$$

where in the last step we have interchanged $(B)_{kj}$ and $(A)_{jk}$ because they are just numbers. Thus (5.9) follows.

Example 5.2.7. Let

$$A = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix},$$

show that

$$(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}.$$

Solution 5.2.7.

$$AB = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 22 \\ -2 & -4 \end{pmatrix}, \quad (AB)^{\mathrm{T}} = \begin{pmatrix} 8 & -2 \\ 22 & -4 \end{pmatrix},$$

$$\widetilde{B} = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}; \quad \widetilde{B}\widetilde{A} = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ 22 & -4 \end{pmatrix}.$$

Thus, $(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}$.

Trace of a Matrix

The trace of square matrix $A = (a_{ij})$ is defined as the sum of its diagonal elements and is denoted by $\operatorname{Tr} A$

$$\operatorname{Tr} A = \sum_{i=1}^{n} a_{ii}.$$

An important theorem about trace is that the trace of the product of a finite number of matrices is invariant under any cyclic permutation of the matrices. We can first prove this theorem for the product of two matrices, and then the rest automatically follow.

Let A be a $n \times m$ matrix and B be a $m \times n$ matrix, then

$$\operatorname{Tr}(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji},$$
$$\operatorname{Tr}(BA) = \sum_{j=1}^{n} (BA)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij}.$$

Since a_{ij} and b_{ij} are just numbers, their order can be reversed. Thus

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA).$$

Notice that the trace is defined only for a square matrix, but A and B do not have to be square matrices as long as their product is a square matrix. The order of AB may be different from the order of BA, yet their traces are the same.

Now

$$\begin{aligned} \operatorname{Tr}(ABC) &= \operatorname{Tr}(A(BC)) = \operatorname{Tr}((BC)A) \\ &= \operatorname{Tr}(BCA) = \operatorname{Tr}(CAB). \end{aligned} \tag{5.12}$$

It is important to note that the trace of the product of a number of matrices is not invariant under any permutation, but only a cyclic permutation of the matrices.

Example 5.2.8. Let

$$A = \begin{pmatrix} 4 & 0 & 6 \\ 5 & 2 & 1 \\ 7 & 8 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 9 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix},$$

show that (a) $\operatorname{Tr}(A+B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$, and (b) $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

Solution 5.2.8.

(a)

$$\begin{aligned}\mathrm{Tr}(A+B) &= \mathrm{Tr}\left\{\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix}+\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix}\right\} = \mathrm{Tr}\begin{pmatrix}5&0&7\\14&3&3\\7&12&4\end{pmatrix}\\ &= 5+3+4 = 12,\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(A)+\mathrm{Tr}(B) &= \mathrm{Tr}\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix}+\mathrm{Tr}\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix}\\ &= (4+2+3)+(1+1+1) = 12.\end{aligned}$$

(b)

$$\mathrm{Tr}(AB) = \mathrm{Tr}\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix}\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}4&24&10\\23&6&10\\79&20&26\end{pmatrix} = 36,$$

$$\mathrm{Tr}(BA) = \mathrm{Tr}\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix}\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}11&8&9\\55&18&61\\27&16&7\end{pmatrix} = 36.$$

Associative Law of Matrix Multiplication

If A, B, and C are three matrices such that the matrix product AB and BC are defined, then

$$(AB)\,C = A\,(BC)\,. \tag{5.13}$$

In other words, it is immaterial which two matrices are multiplied together first. To prove this, let

$$A = (a_{ij})_{m\times n}\,,\quad B = (b_{ij})_{n\times o}\,,\quad C = (c_{ij})_{o\times p}\,.$$

The ijth element of the left-hand side of (5.13) is then

$$\begin{aligned}((AB)C)_{ij} &= \sum_{k=1}^{o}(AB)_{ik}(C)_{kj} = \sum_{k=1}^{o}\left(\sum_{l=1}^{n}(A)_{il}(B)_{lk}\right)(C)_{kj}\\ &= \sum_{k=1}^{o}\sum_{l=1}^{n}a_{il}b_{lk}c_{kj},\end{aligned}$$

while the ijth element of the right-hand side (5.13) is

$$
\begin{aligned}
(A(BC))_{ij} &= \sum_{l=1}^{n} (A)_{il} (BC)_{lj} = \sum_{l=1}^{n} (A)_{il} \sum_{k=1}^{o} (B)_{lk} (C)_{kj} \\
&= \sum_{l=1}^{n} \sum_{k=1}^{o} a_{il} b_{lk} c_{kj}.
\end{aligned}
$$

Clearly $((AB)C)_{ij} = (A(BC))_{ij}$.

Example 5.2.9. Let

$$
A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix},
$$

show that

$$
A\,(BC) = (AB)\,C.
$$

Solution 5.2.9.

$$
\begin{aligned}
A\,(BC) &= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ 16 & 2 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
(AB)\,C &= \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 2 & -1 \\ 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ 16 & 2 \end{pmatrix}.
\end{aligned}
$$

Clearly $A\,(BC) = (AB)\,C$. This is one of the most important properties of matrix algebra.

Distributive Law of Matrix Multiplication

If A, B, and C are three matrices such that the addition $B + C$ and the product AB and BC are defined, then

$$
A(B + C) = AB + AC. \tag{5.14}
$$

To prove this, let

$$
A = (a_{ij})_{m \times n}, \quad B = (b_{ij})_{n \times p}, \quad C = (c_{ij})_{n \times p}, \tag{5.15}
$$

so that the addition $B+C$ and the products AB and AC are defined. The ijth element of the left-hand side of (5.14) is then

$$\begin{aligned}(A(B+C))_{ij} &= \sum_{k=1}^{n}(A)_{ik}(B+C)_{kj} = \sum_{k=1}^{n}(A)_{ik}(B_{kj}+C_{kj}) \\ &= \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}).\end{aligned}$$

The ijth element of the right-hand side of (5.14) is

$$\begin{aligned}(AB+AC)_{ij} &= (AB)_{ij} + (AC)_{ij} \\ &= \sum_{k=1}^{n}(A)_{ik}(B)_{kj} + \sum_{k=1}^{n}(A)_{ik}(C)_{kj} \\ &= \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}).\end{aligned}$$

Thus, (5.14) follows.

Example 5.2.10. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix},$$

show that

$$C\,(A+B) = CA + CB.$$

Solution 5.2.10.

$$\begin{aligned}C\,(A+B) &= \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix}\left[\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\right] \\ &= \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ 21 & 13 \\ 6 & 0 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}CA+CB &= \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 4 \\ 10 & 2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} -2 & -10 \\ 11 & 11 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ 21 & 13 \\ 6 & 0 \end{pmatrix}.\end{aligned}$$

Hence, $C\,(A+B) = CA + CB$.

5.2.4 Determinant of Matrix Product

We have shown in the chapter on determinants that the value of the determinant of the product of two matrices is equal to the product of two determinants. That is, if A and B are square matrices of the same order, then

$$|AB| = |A|\,|B|\,.$$

This relation is of considerable interests. It is instructive to prove it with the properties of matrix products. We will use 2×2 matrices to illustrate the steps of the proof, but it will be obvious that the process is generally valid for all orders.

1. If D is a diagonal matrix, it is easy to show $|DA| = |D|\,|A|$.
   For example, let $D = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$, then
   $$DA = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} d_{11}a_{11} & d_{11}a_{12} \\ d_{22}a_{21} & d_{22}a_{22} \end{pmatrix},$$
   $$|D| = \begin{vmatrix} d_{11} & 0 \\ 0 & d_{22} \end{vmatrix} = d_{11}d_{22},$$
   $$|DA| = \begin{vmatrix} d_{11}a_{11} & d_{11}a_{12} \\ d_{22}a_{21} & d_{22}a_{22} \end{vmatrix} = d_{11}d_{22}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |D|\,|A|\,.$$
2. Any square matrix can be diagonalized by a series of row operations which add a multiple of a row to another row.
   For example, let $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Multiply row 1 by -3 and add it to row 3, the matrix becomes $\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$. Then add row 2 to row 1, we have the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.
3. Each row operation is equivalent to premultiplying the matrix by an elementary matrix obtained from applying the same operation to the identity matrix.
   For example, multiply row 1 by -3 and add it to row 2 of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain the elementary matrix $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$. Multiply this matrix to the left of B,
   $$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}(B) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix},$$
   we get the same result as operating directly on B. The elementary matrix for adding row 2 to row 1 is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Multiplying this matrix to the left of $\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$, we have the diagonal matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

4. Combine the last equations

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Let

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},$$

we can write the equation as

$$EB = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = D.$$

This equation says that the matrix B is diagonalized by the matrix E, which is the product of a series of elementary matrices.

5. Because of the way E is constructed, multiplying E to the left of any matrix M is equivalent to repeatedly adding a multiple of a row to another row of M. From the theory of determinants, we know that these operations do not change the value of the determinant. For example,

$$|EB| = |D| = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2,$$

$$|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

Therefore the determinant of the diagonalized matrix D is equal to the determinant of the original matrix B,

$$|D| = |B|.$$

In fact M can be any matrix, as along as it is compatible,

$$|EM| = |M|.$$

6. Now let $M = BA$,

$$|E(BA)| = |BA|.$$

But

$$|E(BA)| = |(EB)A| = |DA| = |D|\,|A|,$$

since D is diagonal. On the other hand $|D| = |B|$, therefore

$$|BA| = |B|\,|A|.$$

Since $|B|\,|A| = |A|\,|B|$, it follows $|BA| = |AB|$, even though BA may not be equal to AB.

5.2.5 The Commutator

The difference between the two products AB and BA is known as the commutator

$$[A, B] = AB - BA.$$

If in particular, AB is equal to BA, then

$$[A, B] = 0,$$

the two matrices A and B are said to commute with each other.

It follows directly from the definition that:

- $[A, A] = 0$
- $[A, I] = [I, A] = 0$
- $[A, B] = -[B, A]$
- $[A, (B + C)] = [A, B] + [A, C]$
- $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

Example 5.2.11. Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

show that

$$[\sigma_x, \sigma_y] = 2\mathrm{i}\sigma_z, \quad [\sigma_y, \sigma_z] = 2\mathrm{i}\sigma_x, \quad [\sigma_z, \sigma_x] = 2\mathrm{i}\sigma_y.$$

Solution 5.2.11.

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_y\sigma_x = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = 2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\mathrm{i}\sigma_z.$$

Similarly, $[\sigma_y, \sigma_z] = 2\mathrm{i}\sigma_x$ and $[\sigma_z, \sigma_x] = 2\mathrm{i}\sigma_y$.

Example 5.2.12. If a matrix B commutes with a diagonal matrix with no two elements equal to each other, then B must also be a diagonal matrix.

Solution 5.2.12. To prove this, let B commute with a diagonal matrix A of order n, whose elements are

$$\begin{aligned} (A)_{ij} &= a_i \delta_{ij} \\ a_i \neq a_j \quad &\text{if} \quad i \neq j. \end{aligned} \tag{5.16}$$

We are given that

$$AB = BA.$$

Let the elements of B be b_{ij}, we wish to show that $b_{ij} = 0$, unless $i = j$. Taking the ijth element of both sides, we have

$$\sum_{k=1}^{n} (A)_{ik}(B)_{kj} = \sum_{k=1}^{n} (B)_{ik}(A)_{kj}.$$

On using (5.16), this becomes

$$\sum_{k=1}^{n} a_i \delta_{ik} b_{kj} = \sum_{k=1}^{n} b_{ik} a_k \delta_{kj},$$

with the definition of delta function

$$a_i b_{ij} = b_{ij} a_j.$$

This shows

$$(a_i - a_j) b_{ij} = 0.$$

Thus b_{ij} must be all equal to zero for $i \neq j$, since for those cases $a_i \neq a_j$. The only elements of B which can be different from zero are the diagonal elements b_{ii}, proving that B must be a diagonal matrix.

5.3 Systems of Linear Equations

The method of matrix algebra is very useful in solving a system of linear equations. Let $x_1, x_2, \ldots, x_n$ be a set of n unknown variables. An equation which contains first degree of x_i and no products of two or more variables is called a *linear equation*. The most general system of m linear equations in n unknowns can be written in the form

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m. \end{gathered} \tag{5.17}$$

Here the coefficients a_{ij} and the right-hand side terms d_i are supposed to be known constants.

We can regard the variables $x_1, x_2, \ldots, x_n$ as components of the $n \times 1$ column vector $\mathbf{x}$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and the constants $d_1, d_2, \ldots, d_m$ as components of the $m \times 1$ column vector $\mathbf{d}$

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}.$$

The coefficients a_{ij} can be written as elements of the $m \times n$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

With the matrix multiplication defined in Sect. 5.2, (5.17) can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}.$$

If all the components of d are equal to zero, the system is called *homogeneous*. If at least one component of d is not zero, the system is called *nonhomogeneous*. If the system of linear equations is such that the equations are all satisfied simultaneously by at least one set of values of x_i, then it is said to be consistent. The system is said to be inconsistent if the equations are not satisfied simultaneously by any set of values. An inconsistent system has no solution. A consistent system may have an unique solution, or an infinite number of solutions. In the following sections, we will discuss practical ways of finding these solutions, as well as answer the question of existence and uniqueness of the solutions.

5.3.1 Gauss Elimination Method

Two linear systems are equivalent if every solution of either system is a solution of the other. There are three elementary operations that will transform a linear system into another equivalent system:

1. Interchanging two equations
2. Multiplying an equation through by a nonzero number
3. Adding to one equation a multiple of some other equation

That a system is transformed into an equivalent system by the first operation is quite apparent. The reason that the second and third kinds of operations have the same effect is that when the same operations are done on both sides of an equal sign, the equation should remain valid. In fact, these are just the techniques we learned in elementary algebra to solve a set of simultaneous equations. The goal is to transform the set of equations into a simple form so that the solution is obvious. A practical procedure is suggested by the observation that a linear system, whose coefficient matrix is either upper triangular or diagonal, is easy to solve.

For example, the system of equations

$$\begin{aligned} -2x_2 + x_3 &= 8, \\ 2x_1 - x_2 + 4x_3 &= -3, \\ x_1 - x_2 + x_3 &= -2, \end{aligned} \tag{5.18}$$

can be written as

$$\begin{pmatrix} 0 & -2 & 1 \\ 2 & -1 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -2 \end{pmatrix}.$$

Interchange equation 1 and equation 3, the system becomes

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ 2x_1 - x_2 + 4x_3 &= -3 \\ -2x_2 + x_3 &= 8 \end{aligned} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 8 \end{pmatrix},$$

where we have put the matrix equation representing the system right next to it. Multiply equation 1 of the rearranged system by -2 and add to equation 2, we have

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ x_2 + 2x_3 &= 1 \\ -2x_2 + x_3 &= 8 \end{aligned} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix}.$$

Multiply equation 2 of the last system by 2 and add to equation 3

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ x_2 + 2x_3 &= 1 \\ 5x_3 &= 10 \end{aligned} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 10 \end{pmatrix}. \tag{5.19}$$

These four systems of equations are equivalent because they have the same solution. From the last set of equations, it is clear that $x_3 = 2, x_2 = 1 - 2x_3 = -3$, and $x_1 = -2 + x_2 - x_3 = -7$.

This procedure is often referred to as the *Gauss elimination method*, the *echelon method*, or *triangularization*.

Augmented Matrix

To simplify the writing further, we introduce the augmented matrix. The matrix composed of the coefficient matrix plus an additional column whose elements are the nonhomogeneous constants d_i is called the *augmented matrix* of the system. Thus

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & d_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & d_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & d_n \end{array}\right)$$

is the augmented matrix of (5.17). The portion in front of the vertical line is the coefficient matrix. The entire matrix, disregarding the vertical line, is the augmented matrix of the system. Clearly the augmented matrix is just a succinct expression of the linear system.

Instead of operating on the equations of the system, we can just operate on the rows of the augmented matrix with the three *elementary row operations* which consist of:

1. Interchanging of any two rows
2. Multiplying of any row by a nonzero scalar
3. Adding a multiple of a row to another row

Thus we can summarize the Gauss elimination method as using the elementary row operations to reduce the augmented matrix of the original system to an echelon form. A matrix is said to be in echelon form if:

1. The first element in the first row is nonzero.
2. The first $(n-1)$ elements of the nth row are zero, the rest elements may or may not be zero.
3. The first nonzero element of any row appears to the right of the first nonzero element in the row above.
4. As a consequence, if there are rows whose elements are all zero, then they must be at the bottom of the matrix.

Thus we can think of solving the linear system of (5.18) in the above example as reducing the augmented matrix from

$$\left(\begin{array}{ccc|c} 0 & -2 & 1 & 8 \\ 2 & -1 & 4 & -3 \\ 1 & -1 & 1 & -2 \end{array}\right)$$

to the echelon form

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

from which the solution is easily obtained.

Gauss–Jordan Elimination Method

For a large set of linear equations, it is sometimes advantageous to continue the process to reduce the coefficient matrix from the triangular form to a diagonal form. For example, multiply the third row of the last matrix by 1/5

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \tag{5.20}$$

Multiply row 3 by -2 and add to row 2

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Multiply row 3 by -1 and add to row 1:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Add row 2 to row 1:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right),$$

which corresponds to $x_1 = -7, x_2 = -3$, and $x_3 = 2$. This process is known as the *Gauss–Jordan elimination method.*

5.3.2 Existence and Uniqueness of Solutions of Linear Systems

For a linear system of m equations and n unknowns, the order of the coefficient matrix is $m \times n$ that of the augmented matrix is $m \times (n+1)$. If $m < n$, the system is underdetermined. If $m > n$, the system is overdetermined. The most interesting case is $m = n$. In all three cases, we can use Gauss elimination method to reduce the augmented matrix into an echelon form. Once in the echelon form, the problem is either solved, or else shown to be inconsistent. A few examples will make this clear.

Example 5.3.1. Solve the following system of equations:

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, \\ 2x_1 - x_2 + x_3 &= 1, \\ 3x_1 - x_2 + x_3 &= 4. \end{aligned}$$

Solution 5.3.1. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & -1 & 1 & 4 \end{array}\right).$$

Multiply row 1 by -2 and add to row 2:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 3 & -1 & 1 & 4 \end{array}\right).$$

Multiply row 1 by -3 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -4 & 4 & -2 \end{array}\right).$$

Multiply row 2 by $-1/3$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & -2 \end{array}\right).$$

Multiply row 2 by 4 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right).$$

This represents the system of equations

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, \\ x_2 - x_3 &= 1, \\ 0 &= 2. \end{aligned}$$

Since no values of x_1, x_2, and x_3 can make $0 = 2$, the system is inconsistent, and has no solution.

Example 5.3.2. Solve the following system of equations:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 6, \\ 3x_1 - 2x_2 - 8x_3 &= 7, \\ 4x_1 + 5x_2 - 3x_3 &= 17. \end{aligned}$$

Solution 5.3.2. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 3 & -2 & -8 & 7 \\ 4 & 5 & -3 & 17 \end{array}\right).$$

Multiply row 1 by -3 and add to row 2:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & -11 & -11 & -11 \\ 4 & 5 & -3 & 17 \end{array}\right).$$

Multiply row 1 by -4 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & -11 & -11 & -11 \\ 0 & -7 & -7 & -7 \end{array}\right).$$

Multiply row 2 by $-1/11$:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -7 & -7 \end{array}\right).$$

Multiply row 2 by 7 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This represents the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 6, \\ x_2 + x_3 &= 1, \\ 0 &= 0. \end{aligned}$$

This says $x_2 = 1 - x_3$ and $x_1 = 6 - 3x_2 - x_3 = 3 + 2x_3$. The value of x_3 may be assigned arbitrarily, therefore the system has an infinite number of solutions.

Example 5.3.3. Solve the following system of equations:

$$\begin{aligned} x_1 + x_2 &= 2, \\ x_1 + 2x_2 &= 3, \\ 2x_1 + x_2 &= 3. \end{aligned}$$

Solution 5.3.3. The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}\right).$$

Multiply row 1 by -1 and add to row 2:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right).$$

Multiply row 1 by -2 and add to row 3:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right).$$

Add row 2 to row 3:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

The last augmented matrix says

$$\begin{aligned} x_1 + x_2 &= 2, \\ x_2 &= 1, \\ 0 &= 0. \end{aligned}$$

clearly $x_2 = 1$ and $x_1 = 1$. Therefore this system has an unique solution.

To answer questions of existence and uniqueness of solutions of linear systems, it is useful to introduce the concept of the rank of a matrix.

Rank of a Matrix

There are several equivalent definitions for the rank of a matrix. For our purpose, it is most convenient to define the rank of a matrix as the number of nonzero rows in the matrix after it has been transformed into a echelon form by elementary row operations.

In Example 5.3.1, the echelon forms of the coefficient matrix C_{e} and of the augmented matrix A_{e} are, respectively,

$$C_{\mathrm{e}} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\mathrm{e}} = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

In C_{e}, there are two nonzero rows, therefore the rank of the coefficient matrix is 2. In A_{e}, there are three nonzero rows, therefore the rank of the augmented matrix is 3. As we have shown, this system has no solution.

In Example 5.3.2, the echelon forms of these two matrices are

$$C_{\mathrm{e}} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\mathrm{e}} = \begin{pmatrix} 1 & 3 & 1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

They both have only two nonzero rows. Therefore the rank of the coefficient matrix equals the rank of the augmented matrix. They both equal to 2. As we have seen, this system has infinite number of solutions.

In Example 5.3.3, the two echelon forms are

$$C_e = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_e = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both of them have two nonzero rows, therefore the coefficient matrix and the augmented matrix have the same rank of 2. As we have shown, this system has an unique solution.

From the results of these examples, we can make the following observations:

1. A linear system of m equations and n unknowns has solutions if and only if the coefficient matrix and the augmented matrix have the same rank.
2. If the rank of both matrices is r, and $r < n$, the system has infinitely many solutions.
3. If $r = n$, the system has only one solution.

Actually these statements are generally valid for all linear systems regardless of whether $m < n$, $m = n$, or $m > n$.

The most interesting case is $m = n = r$. In that case, the coefficient matrix is a square matrix. The solution of such systems can be obtained from (1) the Cramer's rule discussed in the chapter of determinants, (2) the Gauss elimination method discussed in this section, and (3) the inverse matrix which we will discuss in Sect. 5.4.

5.4 Inverse Matrix

5.4.1 Nonsingular Matrix

The square matrix A is said to be nonsingular if there exists a matrix B such that

$$BA = I,$$

where I is the identity (unit) matrix. If no matrix B exists, then A is said to be singular. The matrix B is the inverse of A and vice versa. The inverse matrix is denoted by A^{-1}

$$A^{-1} = B.$$

The relationship is reciprocal. If B is the inverse of A, then A is the inverse of B. Since

$$BA = A^{-1}A = I, \tag{5.21}$$

applying B^{-1} from the left

$$B^{-1}BA = B^{-1}I.$$

It follows:

$$A = B^{-1}. \tag{5.22}$$

Existence

If A is nonsingular, then determinant $|A| \neq 0$.

Proof. If A is nonsingular, then by definition A^{-1} exists and $AA^{-1} = I$. Thus

$$\left|AA^{-1}\right| = |A| \cdot \left|A^{-1}\right| = |I| .$$

Since $|I| = 1$, neither $|A|$ nor $\left|A^{-1}\right|$ can be zero.

If $|A| \neq 0$, we will show in following sections that A^{-1} can always be found.
□

Uniqueness

The inverse of a matrix, if it exists, is unique. That is, if

$$AB = I,$$
$$AC = I,$$

then

$$B = C.$$

This can be seen as follows. Since $AC = I$, by definition $C = A^{-1}$. It follows that:

$$CA = AC = I.$$

Multiplying this equation from the right by B, we have

$$(CA)B = IB = B.$$

But

$$(CA)B = C(AB) = CI = C.$$

It is clear from the last two equations that $B = C$.

Inverse of Matrix Products

The inverse of the product of a number of matrices, none of which is singular, equals the product of the inverses taken in the reverse order.

Proof. Consider three nonsingular matrices A, B, and C. We will show

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

By definition

$$ABC(ABC)^{-1} = I.$$

Now

$$\begin{aligned} ABC(C^{-1}B^{-1}A^{-1}) &= AB(CC^{-1})B^{-1}A^{-1} \\ &= ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I. \end{aligned}$$

Since the inverse is unique, it follows that:

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

5.4.2 Inverse Matrix by Cramer's Rule

To find A^{-1}, let us consider the set of nonhomogeneous linear equation

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \tag{5.23}$$

written as

$$(A)(x) = (d). \tag{5.24}$$

According to Cramer's rule discussed in the chapter on determinants

$$x_i = \frac{N_i}{|A|},$$

where $|A|$ is the determinant of A and N_i is the determinant

$$N_i = \begin{vmatrix} a_{11} & \dots & a_{1i-1} & d_1 & a_{1i+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2i-1} & d_2 & a_{2i+1} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{ni-1} & d_n & a_{ni+1} & \dots & a_{nn} \end{vmatrix}.$$

Expanding N_i over the ith column, we have

$$x_i = \frac{1}{|A|} \sum_{j=1}^{n} d_j C_{ji}, \tag{5.25}$$

where C_{ji} is the cofactor of jth row and ith column of A.

Now let $A^{-1} = B$, i.e.,

$$A^{-1} = B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

Applying A^{-1} to (5.24) from the left

$$(A^{-1})(A)(x) = (A^{-1})(d),$$

so

$$(x) = (A^{-1})(d),$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Thus

$$x_i = \sum_{j=1}^{n} b_{ij} d_j. \tag{5.26}$$

Compare (5.25) and (5.26), it is clear

$$b_{ij} = \frac{1}{|A|} C_{ji} = \frac{1}{|A|} \widetilde{C}_{ij}.$$

Thus the process of obtaining the inverse of a nonsingular matrix involves the following steps:

(a) Obtain the cofactor of every element of the matrix A and write the matrix of cofactors in the form

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}.$$

(b) Transpose the matrix of cofactors to obtain

$$\widetilde{C} = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}.$$

(c) Divide this by $\det A$ to obtain the inverse of A

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}.$$

Example 5.4.1. Find the inverse of the following matrix by Cramer's rule:

$$A = \begin{pmatrix} -3 & 1 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & -2 \end{pmatrix}.$$

Solution 5.4.1. The nine cofactors of A are

$$C_{11} = \begin{vmatrix} -6 & 5 \\ 2 & -2 \end{vmatrix} = 2, \quad C_{12} = -\begin{vmatrix} 15 & 5 \\ -5 & -2 \end{vmatrix} = 5, \quad C_{13} = \begin{vmatrix} 15 & -6 \\ -5 & 2 \end{vmatrix} = 0,$$

$$C_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} -3 & -1 \\ -5 & -2 \end{vmatrix} = 1, \quad C_{23} = -\begin{vmatrix} -3 & 1 \\ -5 & 2 \end{vmatrix} = 1,$$

$$C_{31} = \begin{vmatrix} 1 & -1 \\ -6 & 5 \end{vmatrix} = -1, \quad C_{32} = -\begin{vmatrix} -3 & -1 \\ 15 & 5 \end{vmatrix} = 0, \quad C_{33} = \begin{vmatrix} -3 & 1 \\ 15 & -6 \end{vmatrix} = 3.$$

The value of the determinant of A can be obtained from the Laplacian expansion over any row or any column. For example, over the first column

$$|A| = -3C_{11} + 15C_{21} - 5C_{31} = -6 + 0 + 5 = -1.$$

So the inverse exists. The matrix of cofactors C is

$$C = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}.$$

The inverse of A is then obtained by transposing C and dividing it by $\det A$. Therefore

$$A^{-1} = \frac{1}{|A|}\widetilde{C} = \frac{1}{-1}\begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -2 & 0 & 1 \\ -5 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix}. \tag{5.27}$$

It can be directly verified that

$$A^{-1}A = \begin{pmatrix} -2 & 0 & 1 \\ -5 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} -3 & 1 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In literature, the transpose of the cofactor matrix of A is sometimes defined as the adjoint of A, i.e., $\operatorname{adj} A = \widetilde{C}$. However, the name adjoint has another meaning, especially in quantum mechanics. It is often defined as the Hermitian conjugate $A^\dagger$, i.e., $\operatorname{adj} A = A^\dagger$. We will discuss Hermitian matrix in Chap. 6.

For a large matrix, there are more efficient techniques to find the inverse matrix. However, for a 2×2 nonsingular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

one readily obtains from this method

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

This result is simple and useful, It may even be worthwhile to memorize it.

Example 5.4.2. Find the inverse matrices for

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Solution 5.4.2.

$$A^{-1} = \frac{1}{(4-6)} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix},$$

$$R^{-1} = \frac{1}{(\cos^2\theta + \sin^2\theta)} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

One can readily verify

$$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5.4.3 Inverse of Elementary Matrices

Elementary Matrices

An elementary matrix is a matrix that can be obtained from the identity matrix I by an elementary operation. For example, the elementary matrix E_1 obtained from interchanging row 1 and row 2 of the identity matrix of third order is

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, the elementary row operation of interchanging row 1 and row 2 of any matrix A of order $3 \times n$ can be accomplished by premultiplying A by the elementary matrix E_1

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}.$$

The second elementary operation, namely multiplying a row, say row 2, by a scalar k can be accomplished as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Finally, to add the third row k times to the second row, we can proceed in the following way

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} + ka_{31} & a_{22} + ka_{32} \\ a_{31} & a_{32} \end{pmatrix}.$$

Thus, to effect any elementary operation on a matrix A, one may first perform the same elementary operation on an identity matrix to obtain the corresponding elementary matrix. Then premultiply A by the elementary matrix.

Inverse of an Elementary Matrix

Since the elementary matrix is obtained from the elementary operation on the identity matrix, its inverse simply represents the reverse operation. For example, E_1 is obtained from interchanging row 1 and row 2 of the identity matrix I

$$E_1 I = E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$E_1^{-1} E_1 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

E_1^{-1} represents the operation of interchanging row 1 and row 2 of E_1. Thus E_1^{-1} is also given by

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_1.$$

The inverses of the two other kinds of elementary matrices can be obtained in a similar way, namely

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be readily shown by successive elementary operations that

$$E_4 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad E_4^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$$

and

$$E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & n & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -n & -m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5.4.4 Inverse Matrix by Gauss–Jordan Elimination

For a matrix of large order, Cramer's rule is of little practical use. One of the most commonly used methods for inverting a large matrix is the Gauss–Jordan method.

Equation (5.23) can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}, \tag{5.28}$$

or symbolically as

$$(A)(x) = (I)(d). \tag{5.29}$$

If the both sides of this equation is under the same operation, the equation will remain to be valid. We will operate them with the Gauss–Jordan procedure. Each step is an elementary row operation which can be thought as premultiplying (multiplying from the left) both sides by the elementary matrix representing that operation. Thus the entire Gauss–Jordan process is equivalent to multiplying (5.29) by a matrix B which is a product of all the elementary matrices representing the steps of the Gauss–Jordan procedure

$$(B)(A)(x) = (B)(I)(d). \tag{5.30}$$

Since the process reduces the coefficient matrix A to the identity matrix I, so

$$BA = I.$$

Postmultiplying both sides by A^{-1}

$$BAA^{-1} = IA^{-1},$$

we have

$$B = A^{-1}.$$

Therefore when the left-hand side of (5.30) becomes a unit matrix times the column matrix x, the right-hand side of the equation must be equal to the inverse matrix times the column matrix d.

Thus if we want to find the inverse of A, we can first augment A by the identity matrix I, and then use elementary operations to transform this matrix. When the submatrix A is in the form of I, the form assumed of the original identity matrix I must be A^{-1}.

We have found the inverse of

$$A = \begin{pmatrix} -3 & 1 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & -2 \end{pmatrix}$$

in Example 5.4.1 by Cramer's rule. Now let us do the same problem by Gauss–Jordan elimination. First we augment A by the identity matrix I

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 15 & -6 & 5 & 0 & 1 & 0 \\ -5 & 2 & -2 & 0 & 0 & 1 \end{array}\right).$$

Divide the first row by -3, second row by 15, and third row by -5:

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -\frac{6}{15} & \frac{5}{15} & 0 & \frac{1}{15} & 0 \\ 1 & -\frac{2}{5} & \frac{2}{5} & 0 & 0 & -\frac{1}{5} \end{array}\right),$$

leave the first row as it is, subtract the first row from the second row and put it in the second row, and subtract the first row from the third row and put it back in the third row

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{15} & 0 & \frac{1}{3} & \frac{1}{15} & 0 \\ 0 & -\frac{1}{15} & \frac{1}{15} & \frac{1}{3} & 0 & -\frac{1}{5} \end{array}\right),$$

multiply the second and third row by -15

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & -5 & -1 & 0 \\ 0 & 1 & -1 & -5 & 0 & 3 \end{array}\right),$$

leave the second row where it is, subtract it from the third row and put the result back to the third row, and then add $1/3$ of the second row to the first row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & -2 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -5 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 3 \end{array}\right),$$

multiply the third row by -1, and then subtract $1/3$ of it from the first row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -5 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -3 \end{array}\right).$$

Finally we have changed matrix A to the unit matrix I, the original unit matrix on the right side must have changed to A^{-1}, thus

$$A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ -5 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix}.$$

which is the same as (5.27) obtained in Sect. 5.3.

This technique is actually more adapted to modern computers. Computer codes and extensive literature for the Gauss–Jordan elimination method are given in W.H. Press, B.P. Flannery, S.A. Teukolsky,, and W.T. Vetterling, *Numerical Recipes,* 2nd edn. (Cambridge University Press, Cambridge 1992).

Exercises

1. Given two matrices

$$A = \begin{pmatrix} 2 & 5 \\ -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix},$$

find $B - 5A$.

Ans. $\begin{pmatrix} -8 & -25 \\ 12 & -4 \end{pmatrix}$.

2. If A and B are the 2×2 matrices

$$A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix},$$

find the products AB and BA.

Ans. $AB = \begin{pmatrix} 22 & 6 \\ -5 & 5 \end{pmatrix}, \quad BA = \begin{pmatrix} 9 & 11 \\ 2 & 18 \end{pmatrix}$.

3. If

$$A = \begin{pmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -4 \\ 3 & -2 \\ -1 & 1 \end{pmatrix},$$

find AB and BA if they exist.

Ans. $AB = \begin{pmatrix} -5 & -2 \\ 2 & 9 \end{pmatrix}, \quad BA = \begin{pmatrix} 14 & -9 & 0 \\ 12 & -7 & 10 \\ -5 & 3 & -3 \end{pmatrix}$.

4. If

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & -2 \\ -1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 2 & 4 \\ 2 & -1 \end{pmatrix},$$

find AB and BA if they exist.

Ans. $AB = \begin{pmatrix} 2 & 7 \\ 0 & 10 \\ 1 & -7 \\ 8 & -1 \end{pmatrix}$, BA does not exist.

5. If
$$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix},$$
verify the associative law by showing that
$$(AB)\,C = A\,(BC)\,.$$

6. Show that if
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
then
$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$
Hint: $A^n = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]^n$.

7. Given
$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Find all possible products of A, B, C and I, two at a time including squares.
(Note that the products of any two matrices is another matrix in this group. These four matrices form a representation of a mathematical group, known as viergruppe (vier is the German word four).)

8. If
$$A = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix},$$
show that $A^2 = 0$.

9. Find the value of
$$(1\ 2)\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
Ans. 8.

10. Explicitly verify that $(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}$, if
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}.$$

11. Show that matrix A is symmetric, if

$$A = B\widetilde{B}.$$

Hint: $a_{ij} = \sum_k b_{ik}\widetilde{b}_{kj}$.

12. Let $A = \begin{pmatrix} 1 & 3 \\ 5 & 12 \end{pmatrix}$, find a matrix E such that EA is diagonal and $|EA| = |A|$.
Ans. $\begin{pmatrix} -4 & 1 \\ -5 & 1 \end{pmatrix}$.

13. Let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix},$$

explicitly show that

$$AB \neq BA \quad \text{but} \quad |AB| = |BA|.$$

14. Show that if

$$[A, B] \neq 0,$$

then

$$(A - B)(A + B) \neq A^2 - B^2,$$
$$(A + B)^2 \neq A^2 + 2AB + B^2.$$

15. Show that

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

16. Show that

$$\left|A^{-1}\right| = |A|^{-1}.$$

Hint: $AA^{-1} = I, \quad |AB| = |A|\,|B|$.

17. Let

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix},$$

find A^{-1}, B^{-1}, and $(AB)^{-1}$ by Cramer's rule and verify that $(AB)^{-1} = B^{-1}A^{-1}$.

18. Reduce the augmented matrix of the following system to an echelon form and show that this system has no solution

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5, \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2, \\ 4x_1 + 5x_2 + 3x_3 &= 7. \end{aligned}$$

Ans. $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right)$.

19. Solve the following equations by Gauss' elimination

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= -1, \\ 3x_1 - 2x_2 + 2x_3 &= 10, \\ 4x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

Ans. $x_3 = -1, x_2 = -3, x_1 = 2$.

20. Let

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & 2 \\ 4 & 1 & 2 \end{pmatrix},$$

find A^{-1} by Gauss–Jordan elimination. Find x_1, x_2, x_3 from

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} -1 \\ 10 \\ 3 \end{pmatrix},$$

and show that

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 3 \end{pmatrix}.$$

21. Determine the rank of the following matrices:

$$\text{(a)} \begin{pmatrix} 4 & 2 & -1 & 3 \\ 0 & 5 & -1 & 2 \\ 12 & -4 & -1 & 5 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 6 & -1 & 10 & -1 \end{pmatrix}.$$

Ans. (a) 2, (b) 2.

22. Determine if the following systems are consistent. If consistent, is the solution unique?

$$\text{(a)} \quad \begin{aligned} x_1 - x_2 + 3x_3 &= -5, \\ -x_1 + 3x_3 &= 0, \\ 2x_1 + x_2 &= 1. \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x_1 - 2x_2 + 3x_3 &= 0, \\ 2x_1 + 3x_2 - x_3 &= 0, \\ 4x_1 - x_2 + 5x_3 &= 0. \end{aligned}$$

Ans. (a) unique solution, (b) infinite number of solutions.

23. Find the value of λ so that the following linear system has a solution

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2, \\ 3x_1 + 2x_2 + x_3 &= 0, \\ x_1 + x_2 + x_3 &= \lambda. \end{aligned}$$

Ans. $\lambda = 0.5$.

24. Let

$$L^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$|-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |\text{null}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Show that

$$\begin{aligned} L^+ \, |-1\rangle = |0\rangle \,, \quad L^+ \, |0\rangle = |1\rangle \,, \quad L^+ \, |1\rangle = |\text{null}\rangle \,, \\ L^- \, |1\rangle = |0\rangle \,, \quad L^- \, |0\rangle = |-1\rangle \,, \quad L^- \, |-1\rangle = |\text{null}\rangle \,. \end{aligned}$$

6

Eigenvalue Problems of Matrices

Given a square matrix A, to determine the scalars λ and the nonzero column matrix $\mathbf{x}$ which simultaneously satisfy the equation

$$A\mathbf{x} = \lambda\mathbf{x} \tag{6.1}$$

is known as the *eigenvalue* problem (eigen in German means proper). The solution of this problem is intimately connected to the question of whether the matrix can be transformed into a diagonal form.

The eigenvalue problem is of great interests in many engineering applications, such as mechanical vibrations, alternating currents, and rigid body dynamics. It is of crucial importance in modern physics. The whole structure of quantum mechanics is based on the diagonalization of certain type of matrices.

6.1 Eigenvalues and Eigenvectors

6.1.1 Secular Equation

In the *eigenvalue* problem, the value λ is called the *eigenvalue* (*characteristic value*) and the corresponding column matrix $\mathbf{x}$ is called the *eigenvector* (*characteristic vector*). If A is a $n \times n$ matrix, (6.1) is given by

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Since

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda I \mathbf{x},$$

where I is the unit matrix, we can write (6.1) as

$$(A - \lambda I)\,\mathbf{x} = 0. \tag{6.2}$$

This system has nontrivial solutions if and only if the determinant of the coefficient matrix vanishes

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{6.3}$$

The expansion of this determinant yields a polynomial of degree n in λ, which is called *characteristic polynomial* $P(\lambda)$. The equation

$$P(\lambda) = |A - \lambda I| = 0 \tag{6.4}$$

is known as the *characteristic equation* (or *secular equation*). Its n roots are the eigenvalues and will be denoted $\lambda_1, \lambda_2, \ldots, \lambda_n$. They may be real or complex. When one of the eigenvalues is substituted back into (6.2), the corresponding eigenvector $\mathbf{x}(x_1, x_2, \ldots, x_n)$ may be determined. Note that the eigenvectors may be multiplied by any constant and remain a solution of the equation.

We will denote $\mathbf{x}_i$ as the the eigenvector belonging to the eigenvalue λ_i. That is, if

$$P(\lambda_i) = 0,$$

then

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i.$$

If n eigenvalues are all different, we will have n distinct eigenvectors. If two or more eigenvalues are the same, we say that they are degenerate. In some problems, a degenerate eigenvalue may produce only one eigenvector, in other problems a degenerate eigenvalue may produce more than one distinct eigenvectors.

Example 6.1.1. Find the eigenvalues and eigenvectors of A, if

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Solution 6.1.1. The characteristic polynomial of A is

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3,$$

and the secular equation is

$$\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0.$$

Thus the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 3.$$

Let the eigenvector $\mathbf{x}_1$ corresponding to $\lambda_1 = -1$ be $\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$, then $\mathbf{x}_1$ must satisfy

$$\begin{pmatrix} 1-\lambda_1 & 2 \\ 2 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0.$$

This reduces to

$$2x_{11} + 2x_{12} = 0.$$

Thus for this eigenvector $x_{11} = -x_{12}$. That is, $x_{11} : x_{12} = -1 : 1$. Therefore the eigenvector can be written as

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Any constant, positive or negative, times it will also be a solution, but it will not be regarded as another distinct eigenvector. With a similar procedure, we find the eigenvector corresponding to $\lambda_2 = 3$ to be

$$\mathbf{x}_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example 6.1.2. Find the eigenvalues and eigenvectors of A, if

$$A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}.$$

Solution 6.1.2. The characteristic polynomial of A is

$$P(\lambda) = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2,$$

so the secular equation is

$$\lambda^2 - 2\lambda + 2 = 0.$$

Thus the eigenvalues are

$$\lambda = 1 \pm \mathrm{i}.$$

Let $\lambda_1 = 1 + \mathrm{i}$, and the corresponding eigenvector $\mathbf{x}_1$ be $\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$, then $\mathbf{x}_1$ must satisfy

$$\begin{pmatrix} 3-(1+\mathrm{i}) & -5 \\ 1 & -1-(1+\mathrm{i}) \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0,$$

which gives

$$(2-\mathrm{i})x_{11} - 5x_{12} = 0,$$

$$x_{11} - (2+\mathrm{i})x_{12} = 0.$$

The first equation gives

$$x_{11} = \frac{5}{2-\mathrm{i}} x_{12} = \frac{5(2+\mathrm{i})}{4+1} x_{12} = \frac{2+\mathrm{i}}{1} x_{12},$$

which is the same result from the second equation, as it should be. Therefore $\mathbf{x}_1$ can be written as

$$\mathbf{x}_1 = \begin{pmatrix} 2+\mathrm{i} \\ 1 \end{pmatrix}.$$

Similarly, for $\lambda = \lambda_2 = 1 - \mathrm{i}$, the corresponding eigenvector is

$$\mathbf{x}_2 = \begin{pmatrix} 2-\mathrm{i} \\ 1 \end{pmatrix}.$$

So we have an example of a real matrix with complex eigenvalues and complex eigenvectors.

Example 6.1.3. Find the eigenvalues and eigenvectors of A, if

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}.$$

Solution 6.1.3. The characteristic polynomial of A is

$$P(\lambda) = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 21\lambda + 45.$$

The secular equation can be written as

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = (\lambda - 5)(\lambda + 3)^2 = 0.$$

This equation has a single root of 5 and a double root of -3. Let

$$\lambda_1 = 5, \quad \lambda_2 = -3, \quad \lambda_3 = -3.$$

The eigenvector belonging to the eigenvalue of λ_1 must satisfy the equation

$$\begin{pmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = 0.$$

With Gauss' elimination method, this equation can be shown to be equivalent to

$$\begin{pmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = 0,$$

which means

$$\begin{aligned} -7x_{11} + 2x_{12} - 3x_{13} &= 0, \\ x_{12} + 2x_{13} &= 0. \end{aligned}$$

Assign $x_{13} = 1$, then $x_{12} = -2, x_{11} = -1$. So corresponding to $\lambda_1 = 5$, the eigenvector $\mathbf{x}_1$ can be written as

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

Since the eigenvalue of -3 is twofold degenerate, corresponding to this eigenvalue, we may have one or two eigenvectors. Let us express the eigenvector corresponding to the eigenvalue of -3 as $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. It must satisfy the equation

$$\begin{pmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

With Gauss' elimination method, this equation can be shown to be equivalent to

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

which means

$$x_1 + 2x_2 - 3x_3 = 0.$$

We can express x_1 in terms of x_2 and x_3, and there is no restriction on x_2 and x_3. Let $x_2 = c_2$ and $x_3 = c_3$, then $x_1 = -2c_2 + 3c_3$. So we can write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2c_2 + 3c_3 \\ c_2 \\ c_3 \end{pmatrix} = c_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

Since c_2 and c_3 are arbitrary, we can first assign $c_3 = 0$ and get an eigenvector, and then assign $c_2 = 0$ and get another eigenvector. So corresponding to the degenerate eigenvalue $\lambda = -3$, there are two distinct eigenvectors

$$\mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

In this example, we have only two distinct eigenvalues, but we still have three distinct eigenvectors.

Example 6.1.4. Find the eigenvalues and eigenvectors of A, if

$$A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}.$$

Solution 6.1.4. The characteristic polynomial of A is

$$P(\lambda) = \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = -\lambda^3 + 5\lambda^2 - 8\lambda + 4.$$

The secular equation can be written as

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2 = 0.$$

The three eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 2.$$

From the equation for the eigenvector $\mathbf{x}_1$ belonging to the eigenvalue of λ_1

$$\begin{pmatrix} 4-1 & 6 & 6 \\ 1 & 3-1 & 2 \\ -1 & -5 & -2-1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = 0,$$

we obtain the solution

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}.$$

The eigenvector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, corresponding to the twofold degenerate eigenvalue 2, satisfies the equation

$$\begin{pmatrix} 4-2 & 6 & 6 \\ 1 & 3-2 & 2 \\ -1 & -5 & -2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

With Gauss' elimination method, this equation can be shown to be equivalent to

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

which means

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0, \\ 2x_2 + x_3 &= 0. \end{aligned}$$

If we assign $x_3 = -2$, then $x_2 = 1$ and $x_1 = 3$. So

$$\mathbf{x}_2 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}.$$

The two equations above do not allow any other eigenvector which is not just a constant times $\mathbf{x}_2$. Therefore for this 3×3 matrix, there are only two distinct eigenvectors.

Computer Code

It should be noted that for large systems, the eigenvalues and eigenvectors would usually be found with specialized numerical methods (see, for example, G.H. Golub and C.F. Van Loan, *Matrix Computations*, John Hopkins University Press, 1983). There are excellent general purpose computer programs for the efficient and accurate determination of eigensystems (see, for example, B.T. Smith, J.M. Boyle, J. Dongarra, B. Garbow, Y. Ikebe, V.C. Klema, and C.B. Moler, *Matrix Eigensystem Routines: EISPACK Guide*, 2nd edn. Springer-Verlag, 1976).

In addition, eigenvalues and eigenvectors can be found with a simple command in computer packages such as, Maple, Mathematica, MathCad, and MuPAD. These packages are known as *Computer Algebraic Systems*.

This book is written with the software "Scientific WorkPlace," which also provides an interface to MuPAD (before version 5, it also came with Maple).

Instead of requiring the user to adhere to a rigid syntax, the user can use natural mathematical notations. For example, to find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix},$$

all you have to do is (1) type the expression in the math-mode, (2) click on the "Compute" button, (3) click on the "Matrices" button in the pull-down menu, and (4) click on the "Eigenvectors" button in the submenu. The program will return with

$$\text{eigenvectors}: \quad \left\{ \begin{pmatrix} 1 \\ -\frac{1}{3} \\ 1 \end{pmatrix} \right\} \leftrightarrow 1, \quad \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 2.$$

You can ask the program to check the results. For example, you can type

$$\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},$$

and click on the "Compute" button, and then click on the "Evaluate" button. The program will return with

$$\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix},$$

which is of course equal to $2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, showing $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is indeed an eigenvector belonging to eigenvalue 2. The other two eigenvectors can be similarly checked.

Computer Algebraic Systems are wonderful as they are, they must be used with caution. It is not infrequent that the system will return with an answer to a wrong problem without the user knowing it. Therefore answers from these systems should be checked. Computer Algebraic Systems are useful supplements, but they are no substitute for the knowledge of the subject matter.

6.1.2 Properties of Characteristic Polynomial

The characteristic polynomial has several useful properties. To elaborate on them, let us first consider the case of $n = 3$.

$$P(\lambda)=\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}+\begin{vmatrix} -\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}+\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & -\lambda & a_{23} \\ a_{31} & 0 & a_{33}-\lambda \end{vmatrix}$$

$$+\begin{vmatrix} -\lambda & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33}-\lambda \end{vmatrix}+\begin{vmatrix} -\lambda & 0 & a_{13} \\ 0 & -\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & -\lambda \end{vmatrix}+\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & -\lambda & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & -\lambda & 0 \\ a_{31} & 0 & -\lambda \end{vmatrix}$$

$$+\begin{vmatrix} -\lambda & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} -\lambda & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & -\lambda \end{vmatrix}+\begin{vmatrix} -\lambda & 0 & a_{13} \\ 0 & -\lambda & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}+\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$=|A|+\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right)(-\lambda)$$

$$+(a_{11}+a_{22}+a_{33})(-\lambda)^2+(-\lambda)^3. \tag{6.5}$$

Now let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues, so $P(\lambda_1)=P(\lambda_2)=P(\lambda_3)=0$. Since $P(\lambda)$ is a polynomial of degree 3, it follows that:

$$P(\lambda)=(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)=0.$$

Expanding the characteristic polynomial,

$$P(\lambda)=\lambda_1\lambda_2\lambda_3+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)(-\lambda)+(\lambda_1+\lambda_2+\lambda_3)(-\lambda)^2+(-\lambda)^3.$$

Comparison with (6.5) shows

$$\lambda_1+\lambda_2+\lambda_3=a_{11}+a_{22}+a_{33}=\operatorname{Tr} A.$$

This means that the sum of the eigenvalues is equal to the trace of A. This is a very useful relation to check if the eigenvalues are calculated correctly. Furthermore

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

which is the sum of principal minors (minors of the diagonal elements), and

$$\lambda_1\lambda_2\lambda_3 = |A|\,.$$

That the product of all eigenvalues is equal to the determinant of A is also a very useful relation. If A is singular $|A| = 0$, at least one of the eigenvalue must be zero. It follows that the inverse of A exists if and only if none of the eigenvalues of A is zero.

Similar calculations can generalize these relationships for matrices of higher orders.

Example 6.1.5. Find the eigenvalues and the corresponding eigenvectors of the matrix A if

$$A = \begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix}.$$

Solution 6.1.5.

$$\begin{aligned} P(\lambda) &= \begin{pmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{pmatrix} \\ &= \begin{vmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{vmatrix} - \left(\begin{vmatrix} 4 & -1 \\ 8 & -3 \end{vmatrix} + \begin{vmatrix} 5 & -5 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 5 & 7 \\ 0 & 4 \end{vmatrix} \right) \lambda + (5+4-3)\,\lambda^2 - \lambda^3 \\ &= 6 - 11\lambda + 6\lambda^2 - \lambda^3 = (1-\lambda)(2-\lambda)\,(3-\lambda) = 0. \end{aligned}$$

Thus the three eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$$

As a check, the sum of the eigenvalues

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 3 = 6$$

is indeed equal to the trace of A

$$\operatorname{Tr} A = 5 + 4 - 3 = 6.$$

Furthermore, the product of three eigenvalues

$$\lambda_1\lambda_2\lambda_3 = 6$$

is indeed equal to the determinant

$$\begin{vmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{vmatrix} = 6.$$

Let the eigenvector $\mathbf{x}_1$ corresponding to λ_1 be $\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}$, then

$$\begin{pmatrix} 5-\lambda_1 & 7 & -5 \\ 0 & 4-\lambda_1 & -1 \\ 2 & 8 & -3-\lambda_1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 4 & 7 & -5 \\ 0 & 3 & -1 \\ 2 & 8 & -4 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = 0.$$

By Gauss' elimination method, one can readily show that

$$\begin{pmatrix} 4 & 7 & -5 \\ 0 & 3 & -1 \\ 2 & 8 & -4 \end{pmatrix} \Longrightarrow \begin{pmatrix} 4 & 7 & -5 \\ 0 & 3 & -1 \\ 0 & 4.5 & -1.5 \end{pmatrix} \Longrightarrow \begin{pmatrix} 4 & 7 & -5 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the set of equations is reduced to

$$\begin{aligned} 4x_{11} + 7x_{12} - 5x_{13} &= 0, \\ 3x_{12} - x_{13} &= 0. \end{aligned}$$

Only one of the three unknowns can be assigned arbitrary. For example, let $x_{13} = 3$, then $x_{12} = 1$ and $x_{11} = 2$. Therefore corresponding to the eigenvalue $\lambda_1 = 1$, the eigenvector can be written as

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

Similarly, corresponding to $\lambda_2 = 2$ and $\lambda_3 = 3$, the respective eigenvectors are

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

6.1.3 Properties of Eigenvalues

There are other properties related to eigenvalue problems. Taken individually, they are almost self-evident, but collectively they are useful in matrix applications.

- The transpose $\widetilde{A}\left(A^{\mathrm{T}}\right)$ has the same eigenvalues as A.

The eigenvalues of A and A^{T} are, respectively, the solutions of $|A - \lambda I| = 0$ and $\left|A^{\mathrm{T}} - \lambda I\right| = 0$. Since $A^{\mathrm{T}} - \lambda I = (A - \lambda I)^{\mathrm{T}}$ and the determinant of a matrix is equal to the determinant of its transpose

$$|A - \lambda I| = \left|(A - \lambda I)^{\mathrm{T}}\right| = \left|A^{\mathrm{T}} - \lambda I\right|,$$

the secular equations of A and A^{T} are identical. Therefore they have the same set of eigenvalues.

- If A is either upper or lower triangular, then the eigenvalues are the diagonal elements.

  Let $|A - \lambda I| = 0$ be

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ 0 & a_{22}-\lambda & \dots & a_{2n} \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & 0 & a_{nn}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0.$$

Clearly $\lambda = a_{11}, \lambda = a_{22}, \dots, \lambda = a_{nn}$.

- If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of the inverse A^{-1} are $1/\lambda_1, 1/\lambda_2, 1/\lambda_3, \dots, 1/\lambda_n$.

  Multiplying the equation $A\mathbf{x} = \lambda\mathbf{x}$ from the left by A^{-1}

$$A^{-1}A\mathbf{x} = A^{-1}\lambda\mathbf{x} = \lambda A^{-1}\mathbf{x}$$

and using $A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x}$, we have $\mathbf{x} = \lambda A^{-1}\mathbf{x}$. Thus

$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}.$$

- If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \dots, \lambda_n^m$.

  Since $A\mathbf{x} = \lambda\mathbf{x}$, it follows:

$$A^2\mathbf{x} = A(A\mathbf{x}) = A\lambda\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}.$$

Similarly

$$A^3\mathbf{x} = \lambda^3\mathbf{x}, \dots, A^m\mathbf{x} = \lambda^m\mathbf{x}.$$

6.2 Some Terminology

As we have seen that for a $n \times n$ square matrix, the eigenvalues may or may not be real numbers. If the eigenvalues are degenerate, we may or may not have n distinct eigenvectors.

However, there is a class of matrices, known as *hermitian matrices*, the eigenvalues of which are always real. A $n \times n$ hermitian matrix will always have n distinct eigenvectors.

To facilitate the discussion of these and other properties of matrices, we will first introduce the following terminology.

6.2.1 Hermitian Conjugation

Complex Conjugation

If $A = (a_{ij})_{m\times n}$ is an arbitrary matrix whose elements may be complex numbers, the complex conjugate matrix denoted by A^* is also a matrix of order $m \times n$ with every element of which is the complex conjugate of the corresponding element of A, i.e.,

$$(A^*)_{ij} = a_{ij}^*.$$

It is clear that

$$(cA)^* = c^* A^*.$$

Hermitian Conjugation

When the two operations of complex conjugation and transposition are carried out one after another on a matrix, the resulting matrix is called the *hermitian conjugate* of the original matrix and is denoted by $A^\dagger$, called A dagger. Mathematicians also refer to $A^\dagger$ as the adjoint matrix. The order of the two operation is immaterial. Thus

$$A^\dagger = (A^*)^{\mathrm{T}} = (\widetilde{A})^*. \tag{6.6}$$

For example, if

$$A = \begin{pmatrix} (6+\mathrm{i}) & (1-6\mathrm{i}) & 1 \\ (3+\mathrm{i}) & 4 & 3\mathrm{i} \end{pmatrix}, \tag{6.7}$$

then

$$A^\dagger = (A^*)^{\mathrm{T}} = \begin{pmatrix} (6-\mathrm{i}) & (1+6\mathrm{i}) & 1 \\ (3-\mathrm{i}) & 4 & -3\mathrm{i} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} (6-\mathrm{i}) & (3-\mathrm{i}) \\ (1+6\mathrm{i}) & 4 \\ 1 & -3\mathrm{i} \end{pmatrix}, \tag{6.8}$$

$$A^\dagger = (\widetilde{A})^* = \begin{pmatrix} (6+\mathrm{i}) & (3+\mathrm{i}) \\ (1-6\mathrm{i}) & 4 \\ 1 & 3\mathrm{i} \end{pmatrix}^* = \begin{pmatrix} (6-\mathrm{i}) & (3-\mathrm{i}) \\ (1+6\mathrm{i}) & 4 \\ 1 & -3\mathrm{i} \end{pmatrix}. \tag{6.9}$$

Hermitian Conjugate of Matrix Products

We have shown in Chap. 5 that the transpose of the product of two matrices is equal to the product of the transposed matrices taken in reverse order. This leads directly to the fact that

$$(AB)^\dagger = B^\dagger A^\dagger,$$

since

$$(AB)^\dagger = (A^* B^*)^{\mathrm{T}} = \widetilde{B}^* \widetilde{A}^* = B^\dagger A^\dagger. \tag{6.10}$$

6.2.2 Orthogonality

Inner Product

If $\mathbf{a}$ and $\mathbf{b}$ are two column vectors of the same order n, the inner product (or *scalar product*) is defined as $\mathbf{a}^{\dagger}\mathbf{b}$. The hermitian conjugate of the column vector is a row vector

$$\mathbf{a}^{\dagger} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^{\dagger} = \begin{pmatrix} a_1^* \; a_2^* \cdots a_n^* \end{pmatrix},$$

therefore the inner product is one number

$$\mathbf{a}^{\dagger}\mathbf{b} = \begin{pmatrix} a_1^* \; a_2^* \cdots a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} a_k^* b_k.$$

There are two other commonly used notations for the inner product.

The notation most often used in quantum mechanics is the bracket notation of Dirac. The row and column vectors are, respectively, defined as the bra and ket vectors. Thus we may write the column vector

$$\mathbf{b} = |\,\mathbf{b}\rangle$$

as the ket vector, and the row vector

$$\mathbf{a}^{\dagger} = \langle \mathbf{a}|$$

as the bra vector. The inner product of two vectors is then represented by

$$\langle \mathbf{a}|\mathbf{b}\rangle = \mathbf{a}^{\dagger}\mathbf{b}.$$

Notice that for any scalar c,

$$\langle \mathbf{a}|c\mathbf{b}\rangle = c\,\langle \mathbf{a}|\mathbf{b}\rangle,$$

whereas

$$\langle c\mathbf{a}|\mathbf{b}\rangle = c^*\,\langle \mathbf{a}|\mathbf{b}\rangle.$$

Another notation that is often used is the parenthesis notation:

$$(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\dagger}\mathbf{b} = \langle \mathbf{a}|\mathbf{b}\rangle.$$

If A is a matrix, then

$$(\mathbf{a}, A\mathbf{b}) = \left(A^{\dagger}\mathbf{a}, \mathbf{b}\right)$$

is an identity, since

$$\left(A^{\dagger}\mathbf{a},\mathbf{b}\right)=\left(A^{\dagger}\mathbf{a}\right)^{\dagger}\mathbf{b}=\mathbf{a}^{\dagger}\left(A^{\dagger}\right)^{\dagger}\mathbf{b}=\mathbf{a}^{\dagger}A\mathbf{b}=(\mathbf{a},A\mathbf{b})\,.$$

Thus if

$$(\mathbf{a},A\mathbf{b})=(A\mathbf{a},\mathbf{b})\,,$$

then A is hermitian. Mathematicians refer to the relation $A^{\dagger}=A$ as *self-adjoint*.

Orthogonality

Two vectors **a** and **b** are said to be orthogonal if and only if

$$\mathbf{a}^{\dagger}\mathbf{b}=0.$$

Note that in three-dimensional real space

$$\mathbf{a}^{\dagger}\mathbf{b}=\sum_{k=1}^{n}a_k^{*}b_k^{*}=a_1b_1+a_2b_2+a_3b_3$$

is just the dot product of **a** and **b**. It is well known in vector analysis that if the dot product of two vectors is equal to zero, then they are perpendicular.

Length of a Complex Vector

If we adopt this definition of the scalar product of two complex vectors, then we have a natural definition of the length of a complex vector in a n-dimensional space. The length $\|\mathbf{x}\|$ of a complex vector $\mathbf{x}$ is taken to be

$$\|\mathbf{x}\|^2=\mathbf{x}^{\dagger}\mathbf{x}=\sum_{k=1}^{n}a_k^{*}a_k=\sum_{k=1}^{n}|a_k|^2\,.$$

6.2.3 Gram–Schmidt Process

Linear Independence

The set of vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$ is linearly independent if and only it

$$\sum_{i=1}^{n}a_i\mathbf{x}_i=0$$

implies every $a_i=0$. Otherwise the set is linearly dependent.

Let us test the three vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

for linear independence. The question is if we can find a set of a_i, not all zero such that

$$\sum_{i=1}^{3} a_i \mathbf{x}_i = a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_3 \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly this requires $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. Therefore these three vectors are linearly independent.

Note that linear independence or dependence is a property of the set as a whole, not of the individual vectors.

It is obvious that if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ represent three noncoplanar three-dimensional vectors, they are linearly independent.

Gram–Schmidt Process

Given any n linearly independent vectors, one can construct from their linear combinations a set of n mutually orthogonal unit vectors.

Let the given linearly independent vectors be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Define

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

to be the first unit vector. Now define

$$\mathbf{u}_2' = \mathbf{x}_2 - (\mathbf{x}_2, \mathbf{u}_1)\, \mathbf{u}_1.$$

The inner product of $\mathbf{u}_2'$ and $\mathbf{u}_1$ is equal to zero,

$$(\mathbf{u}_2', \mathbf{u}_1) = (\mathbf{x}_2, \mathbf{u}_1) - (\mathbf{x}_2, \mathbf{u}_1)\,(\mathbf{u}_1, \mathbf{u}_1) = 0,$$

since $(\mathbf{u}_1, \mathbf{u}_1) = 1$. This shows $\mathbf{u}_2'$ is orthogonal to $\mathbf{u}_1$.

We can normalize $\mathbf{u}_2'$

$$\mathbf{u}_2 = \frac{\mathbf{u}_2'}{\|\mathbf{u}_2'\|}$$

to obtain the second unit vector $\mathbf{u}_2$ which is orthogonal to $\mathbf{u}_1$.

We can continue this process by defining

$$\mathbf{u}_k' = \mathbf{x}_k - \sum_{i=1}^{k-1} (\mathbf{x}_k, \mathbf{u}_i)\, \mathbf{u}_i,$$

and

$$\mathbf{u}_k = \frac{\mathbf{u}'_k}{\|\mathbf{u}'_k\|}.$$

When all $\mathbf{x}_k$ are used up, we will have n unit vectors $u_1, u_2, \ldots, u_k$ orthogonal to each other. They are called an *orthonormal set*. This procedure is known as Gram–Schmidt process.

6.3 Unitary Matrix and Orthogonal Matrix

6.3.1 Unitary Matrix

If the square matrix U satisfies the condition

$$U^\dagger U = I,$$

then U is called unitary. The n columns in a unitary matrix can be considered as n column vectors in an orthonormal set.

In other words, if

$$\mathbf{u}_1 = \begin{pmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1n} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} u_{21} \\ u_{22} \\ \vdots \\ u_{2n} \end{pmatrix}, \ldots, \mathbf{u}_n = \begin{pmatrix} u_{n1} \\ u_{n2} \\ \vdots \\ u_{nn} \end{pmatrix},$$

and

$$\mathbf{u}_i^\dagger \mathbf{u}_j = \begin{pmatrix} u_{i1}^* & u_{i2}^* & \cdots & u_{in}^* \end{pmatrix} \begin{pmatrix} u_{j1} \\ u_{j2} \\ \vdots \\ u_{jn} \end{pmatrix} = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases},$$

then

$$U = \begin{pmatrix} u_{11} & u_{21} & \ldots & u_{n1} \\ u_{12} & u_{22} & \ldots & u_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1n} & u_{n2} & \ldots & u_{nn} \end{pmatrix}$$

is unitary. This is because

$$U^\dagger = \begin{pmatrix} u_{11}^* & u_{12}^* & \ldots & u_{1n}^* \\ u_{21}^* & u_{22}^* & \ldots & u_{2n}^* \\ \vdots & \vdots & \ldots & \vdots \\ u_{n1}^* & u_{n2}^* & \ldots & u_{nn}^* \end{pmatrix}$$

therefore

$$U^{\dagger}U = \begin{pmatrix} u_{11}^* & u_{12}^* & \dots & u_{1n}^* \\ u_{21}^* & u_{22}^* & \dots & u_{2n}^* \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1}^* & u_{n2}^* & \dots & u_{nn}^* \end{pmatrix} \begin{pmatrix} u_{11} & u_{21} & \dots & u_{n1} \\ u_{12} & u_{22} & \dots & u_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1n} & u_{2n} & \dots & u_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Multiply U^{-1} from the right, we have

$$U^{\dagger}UU^{-1} = IU^{-1}.$$

It follows that hermitian conjugate of a unitary matrix is its inverse, i.e.,

$$U^{\dagger} = U^{-1}.$$

6.3.2 Properties of Unitary Matrix

- Unitary transformations leave lengths of vectors invariant.

Let

$$\mathbf{a} = U\mathbf{b}, \quad \text{so} \quad \mathbf{a}^{\dagger} = \mathbf{b}^{\dagger}U^{\dagger},$$

and

$$\|\mathbf{a}\|^2 = \mathbf{a}^{\dagger}\mathbf{a} = \mathbf{b}^{\dagger}U^{\dagger}U\mathbf{b}.$$

Since

$$U^{\dagger}U = U^{-1}U = I,$$

it follows:

$$\|\mathbf{a}\|^2 = \mathbf{a}^{\dagger}\mathbf{a} = \mathbf{b}^{\dagger}\mathbf{b} = \|\mathbf{b}\|^2.$$

Thus the length of the initial vector is equal to the length of the transformed vector.

- The absolute value of the eigenvalues of an unitary matrix is equal to one.

Let $\mathbf{x}$ be a nontrivial eigenvector of the unitary matrix U belonging to the eigenvalue λ

$$U\mathbf{x} = \lambda\mathbf{x}.$$

Take the hermitian conjugate of both sides

$$\mathbf{x}^{\dagger}U^{\dagger} = \lambda^*\mathbf{x}^{\dagger}.$$

Multiply the last two equations

$$\mathbf{x}^{\dagger}U^{\dagger}U\mathbf{x} = \lambda^{*}\mathbf{x}^{\dagger}\lambda\mathbf{x}.$$

Since $U^{\dagger}U = I$ and $\lambda^{*}\lambda = |\lambda|^{2}$, it follows:

$$\mathbf{x}^{\dagger}\mathbf{x} = |\lambda|^{2}\,\mathbf{x}^{\dagger}\mathbf{x}.$$

Therefore

$$|\lambda|^{2} = 1.$$

In other words, the eigenvalues of a unitary matrix must be on the unit circle in the complex plane centered at the origin.

6.3.3 Orthogonal Matrix

If the elements of an unitary matrix are all real, the matrix is known as an *orthogonal matrix.* Thus the properties of unitary matrices are also properties of orthogonal matrices. In addition,

- The determinant of an orthogonal matrix is equal to either positive one or negative one.

  If A is a real square matrix, then by definition

$$A^{\dagger} = \widetilde{A}^{*} = \widetilde{A}.$$

If, in addition, A is unitary, $A^{\dagger} = A^{-1}$, then

$$\widetilde{A} = A^{-1}.$$

Thus

$$A\widetilde{A} = I. \tag{6.11}$$

Since the determinant of A is equal to the determinant of $\widetilde{A}$, so

$$|A\widetilde{A}| = |A||\widetilde{A}| = |A|^{2}.$$

But

$$|A\widetilde{A}| = |I| = 1,$$

therefore

$$|A|^{2} = 1.$$

Thus, the determinant of an orthogonal matrix is either $+1$ or -1.

Very often (6.11) is used as the definition of an orthogonal matrix. That is, a square real matrix A satisfying the relation expressed in (6.11) is called an orthogonal matrix. This is equivalent to the statement "that the inverse of an orthogonal matrix is equal to its transpose."

Written in terms of its elements, (6.11) is given by

$$\sum_{j=1} a_{ij}\widetilde{a}_{jk} = \sum_{j=1} a_{ij}a_{kj} = \delta_{ik} \tag{6.12}$$

for any i and any j. Similarly, $\widetilde{A}A = I$ can be expressed as

$$\sum_{j=1} \widetilde{a}_{ij}a_{jk} = \sum_{j=1} a_{ji}a_{jk} = \delta_{ik}. \tag{6.13}$$

However, (6.13) is not independent of (6.12), since $A\widetilde{A} = \widetilde{A}A$. If one set of conditions is valid, the other set must also be valid.

Put in words, these conditions mean that the sum of the products of the corresponding elements of two distinct columns (or rows) of an orthogonal matrix is zero, while the sum of the squares of the elements of any column (or row) is equal to unity. If we regard the n columns of the matrix as n real vectors, this means that these n column vectors are orthogonal and normalized. Similarly, all the rows of an orthogonal matrix are orthonormal.

6.3.4 Independent Elements of an Orthogonal Matrix

An nth order square matrix has n^2 elements. For an orthogonal matrix, not all these elements are independent of each other, because there are certain conditions they must satisfy. First, there are n conditions for each column to be normalized. Then there are $n(n-1)/2$ conditions for each column to be orthogonal to any other column. Therefore the number of independent parameters of an orthogonal matrix is

$$n^2 - [n + n(n-1)/2] = n(n-1)/2.$$

In other words, an nth order orthogonal matrix can be fully characterized by $n(n-1)/2$ independent parameters.

For $n = 2$, the number of independent parameters is one. This is illustrated as follows.

Consider an arbitrary orthogonal matrix of order 2

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

The fact that each column is normalized leads to

$$a^2 + b^2 = 1, \tag{6.14}$$

$$c^2 + d^2 = 1. \tag{6.15}$$

Furthermore, the two columns are orthogonal

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd = 0. \tag{6.16}$$

The general solution of (6.14) is $a = \cos\theta$, $b = \sin\theta$, where θ is a scalar. Similarly, (6.15) can be satisfied, if we choose $c = \cos\phi$, $d = \sin\phi$, where ϕ is another scalar. On the other hand, (6.16) requires

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta - \phi) = 0,$$

therefore

$$\phi = \theta \pm \frac{\pi}{2}.$$

Thus, the most general orthogonal matrix of order 2 is

$$A_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad A_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{6.17}$$

Every orthogonal matrix of order 2 can be expressed in this form with some value of θ. Clearly the determinant of A_1 is equal to $+1$ and that of A_2, -1.

6.3.5 Orthogonal Transformation and Rotation Matrix

The fact that in real space, orthogonal transformation preserving the length of a vector suggests that the orthogonal matrix is associated with rotation of vectors. In fact the orthogonal matrix is related to two kinds of rotations in space. First it can be thought as an operator which rotates a vector. This is often called *active transformation*. Secondly, it can be thought as the transformation matrix when the coordinate axes of the reference system are rotated. This is also referred as *passive transformation*.

First let us consider the vectors shown in Fig. 6.1a. The x and y components of the vector $\mathbf{r}_1$ are given by $x_1 = r\cos\varphi$ and $y_1 = r\sin\varphi$, where r is the length of the vector. Now let us rotate the vector counterclockwise by an angle θ, so that $x_2 = r\cos(\varphi + \theta)$ and $y_2 = r\sin(\varphi + \theta)$. Using trigonometry, we can write

$$\begin{aligned} x_2 &= r\cos(\varphi + \theta) = r\cos\varphi\cos\theta - r\sin\varphi\sin\theta = x_1\cos\theta - y_1\sin\theta, \\ y_2 &= r\sin(\varphi + \theta) = r\sin\varphi\cos\theta + r\cos\varphi\sin\theta = y_1\cos\theta + x_1\sin\theta. \end{aligned}$$

We can display the set of coefficients in the form of

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

It is seen that the coefficient matrix is the orthogonal matrix A_1 of (6.17). Therefore the orthogonal matrix with determinant equal to $+1$ is also called

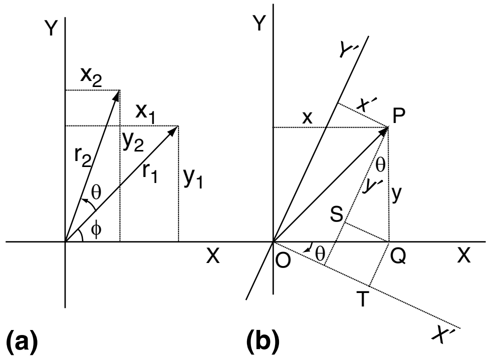

Fig. 6.1. Two interpretations of the orthogonal matrix A_1 whose determinant is equal to +1. (**a**) As a operator, it rotates the vector $\mathbf{r}_1$ to $\mathbf{r}_2$ without changing its length. (**b**) As the transformation matrix between the coordinates of the tip of a fixed vector when the coordinate axes are rotated. Note that the rotation in (**b**) is in the opposite direction as in (**a**)

rotation matrix. It rotates the vector from $\mathbf{r}_1$ to $\mathbf{r}_2$ without changing the magnitude.

The second interpretation of rotation matrix is as follows. Let P be the tip of a fixed vector. The coordinates of P are (x, y) in a particular rectangular coordinate system. Now the coordinate axes are rotated clockwise by an angle θ as shown in Fig. 6.1b. The coordinates of P in the rotated system become (x', y'). From the geometry in Fig. 6.1b, it is clear that

$$\begin{aligned} x' &= OT - SQ = OQ\cos\theta - PQ\sin\theta = x\cos\theta - y\sin\theta, \\ y' &= QT + PS = OQ\sin\theta + PQ\cos\theta = x\sin\theta + y\cos\theta, \end{aligned}$$

or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that the matrix involved is again the orthogonal matrix A_1. However, this time A_1 is the transformation matrix between the coordinates of the tip of a fixed vector when the coordinate axes are rotated.

The equivalence between these two interpretations might be expected, since the relative orientation between the vector and coordinate axes is the same whether the vector is rotated counterclockwise by an angle θ, or the coordinate axes are rotated clockwise by the same angle.

Next, let us consider the orthogonal matrix A_2, the determinant of which is equal to -1. The matrix A_2 can be expressed as

$$A_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

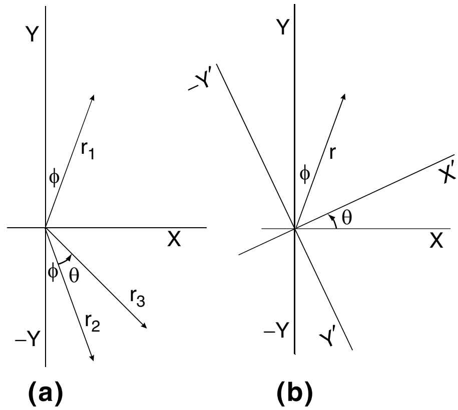

Fig. 6.2. Two interpretations of the orthogonal matrix A_2 whose determinant is -1. (**a**) As an operator, it flips the vector $\mathbf{r}_1$ to $\mathbf{r}_2$ symmetrically with respect to X-axis, and then rotates $\mathbf{r}_2$ to $\mathbf{r}_3$. (**b**) As the transformation matrix between the tip of a fixed vector when the Y-axis is inverted and then the coordinate axes are rotated. Note that the rotation in (**b**) seems to be in the same direction as in (**a**)

The transformation

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

gives

$$x_2 = x_1, \quad y_2 = -y_1.$$

Clearly this corresponds to the reflection of the vector with respect to the X-axis. Therefore A_2 can be considered as an operator which first symmetrically flips the vector $\mathbf{r}_1$ over the X-axis and then rotates it to $\mathbf{r}_3$ as shown in Fig. 6.2a.

In terms of coordinate transformation, one can show that (x', y') in the equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

represents the new coordinates of the tip of a fixed vector after the Y-axis is inverted and the whole coordinate axes are rotated by an angle θ, as shown in Fig. 6.2b. In this case one has to be careful about the sign of the angle. The sign convention is that a counterclockwise rotation is positive and a clockwise rotation is negative. However, after the Y-axis is inverted as in Fig. 6.2b, a negative rotation (rotating from the direction of the positive X-axis toward the negative of the Y-axis) appears to be counterclockwise. This is why in Fig. 6.1a, b, the vector and the coordinate axes are rotating in the opposite direction, whereas in Fig. 6.2a, b, they seem to rotate in the same direction.

So far we have used rotations in two dimensions as examples. However, the conclusions that orthogonal matrix whose determinant equals to $+1$ represents pure rotation, and orthogonal matrix whose determinant is equal to -1

represents a reflection followed by a rotations are generally valid in higher-dimensional space. In the chapter on vector transformation, we will have a more detailed discussion.

6.4 Diagonalization

6.4.1 Similarity Transformation

If A is a $n \times n$ matrix and $\mathbf{u}$ is a $n \times 1$ column vector, then $A\mathbf{u}$ is another $n \times 1$ column vector. The equation

$$A\mathbf{u} = \mathbf{v} \tag{6.18}$$

represents a linear transformation. Matrix A acts as a linear operator sending vector $\mathbf{u}$ to vector $\mathbf{v}$. Let

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

where u_i and v_i are, respectively, the ith components of $\mathbf{u}$ and $\mathbf{v}$ in the n-dimensional space. These components are, of course, measured in a certain coordinate system (reference frame). Let the unit vectors, $\mathbf{e}_i$, known as bases, along the coordinate axes of this system be

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

then

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + u_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{n} u_i \mathbf{e}_i. \tag{6.19}$$

Suppose there is another coordinate system, which we designate as the prime system. Measured in that system, the components of $\mathbf{u}$ and $\mathbf{v}$ become

$$\begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \mathbf{u}', \quad \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{pmatrix} = \mathbf{v}'. \tag{6.20}$$

We emphasize that $\mathbf{u}$ and $\mathbf{u}'$ are the same vector except measured in two different coordinate systems. The symbol $\mathbf{u}'$ does not mean a vector different from $\mathbf{u}$, it simply represents the collection of components of $\mathbf{u}$ in the prime system as shown in (6.20). Similarly, $\mathbf{v}$ and $\mathbf{v}'$ are the same vectors. We can find these components if we know the components of $\mathbf{e}_i$ in the prime system.

In (6.19)

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_n\mathbf{e}_n,$$

the u_i' are just numbers which are independent of the coordinate system. To find the components of $\mathbf{u}$ in the prime system, we only need to express $\mathbf{e}_i$ in the prime system.

Let $\mathbf{e}_i$ measured in the prime system be

$$\mathbf{e}_1 = \begin{pmatrix} s_{11} \\ s_{21} \\ \vdots \\ s_{n1} \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} s_{12} \\ s_{22} \\ \vdots \\ s_{n2} \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} s_{1n} \\ s_{2n} \\ \vdots \\ s_{nn} \end{pmatrix},$$

then the components of $\mathbf{u}$ measured in the prime system can be written as

$$\begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = u_1 \begin{pmatrix} s_{11} \\ s_{21} \\ \vdots \\ s_{n1} \end{pmatrix} + u_2 \begin{pmatrix} s_{12} \\ s_{22} \\ \vdots \\ s_{n2} \end{pmatrix} + \cdots + u_n \begin{pmatrix} s_{1n} \\ s_{2n} \\ \vdots \\ s_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} u_1 s_{11} + u_2 s_{12} + \cdots + u_n s_{1n} \\ u_1 s_{21} + u_2 s_{22} + \cdots + u_n s_{2n} \\ \vdots \\ u_1 s_{n1} + u_2 s_{n2} + \cdots + u_n s_{nn} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ldots & s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{n1} & s_{n2} & \ldots & s_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

This equation can be written in the form

$$\mathbf{u}' = T\mathbf{u}, \tag{6.21}$$

where

$$T = \begin{pmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ldots & s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{n1} & s_{n2} & \ldots & s_{nn} \end{pmatrix}.$$

It is clear from this analysis that the transformation matrix between the vector components in two coordinate systems is the same for all vectors, since it depends only on the transformation of the basis vectors in the two reference frames. Therefore $\mathbf{v}'$ and $\mathbf{v}$ are also related to the same transformation matrix T,

$$\mathbf{v}' = T\mathbf{v}. \tag{6.22}$$

The operation of sending $\mathbf{u}$ to $\mathbf{v}$, expressed in the original system is given by $A\mathbf{u} = \mathbf{v}$. Let the same operation expressed in the prime system be

$$A'\mathbf{u}' = \mathbf{v}'.$$

Since $\mathbf{u}' = T\mathbf{u}$ and $\mathbf{v}' = T\mathbf{v}$,

$$A'T\mathbf{u} = T\mathbf{v}.$$

Multiply both sides by the inverse of T from the left,

$$T^{-1}A'T\mathbf{u} = T^{-1}T\mathbf{v} = \mathbf{v}.$$

Since $A\mathbf{u} = \mathbf{v}$, if follows that:

$$A = T^{-1}A'T. \tag{6.23}$$

If we multiply this equation by T from the left and by T^{-1} from the right, we have

$$TAT^{-1} = A'.$$

What we have found is that as long as we know the relationship between the coordinate axes of two reference frames, not only we can transform a vector from one reference frame to the other, but we can also transform a matrix representing a linear operator from one reference frame to the other.

In general, if there exits a nonsingular matrix T such that $T^{-1}AT = B$ for any two square matrices A and B of the same order, then A and B are called *similar matrices*, and the transformation from A to B is called *similarity transformation.*

If two matrices are related by a similarity transformation, then they represent the same linear transformation in two different coordinate systems.

If the rectangular coordinate axes in the prime system are generated through a rotation from the original system, then T is an orthogonal matrix as discussed in the Sect. 6.3. In that case $T^{-1} = \widetilde{T}$, and the similarity transformation can be written as $\widetilde{T}AT$. If we are working in the complex space, the transformation matrix is unitary, and the similarity transformation can be written as $T^{\dagger}AT$. Both of these transformations are known as unitary similarity transformation.

A matrix that can be brought to diagonal form by a similarity transformation is said to be *diagonalizable.* Whether a matrix is diagonalizable and how to diagonalize it are very important questions in the theory of linear transformation. Not only because it is much more convenient to work with diagonal matrix, but also because it is of fundamental importance in the structure of quantum mechanics. In the following sections, we will answer these questions.

6.4.2 Diagonalizing a Square Matrix

The eigenvectors of the matrix A can be used to form another matrix S in such a way that $S^{-1}AS$ becomes a diagonal matrix. This process often greatly simplifies a physical problem by a better choice of variables.

If A is a square matrix of order n, the eigenvalues λ_i and eigenvectors $\mathbf{x}_i$ satisfy the equation

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i \tag{6.24}$$

for $i = 1, 2, \ldots, n$. Each eigenvector is a column matrix with n elements

$$\mathbf{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}, \ldots, \mathbf{x}_n = \begin{pmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nn} \end{pmatrix}.$$

Each of the n equations of (6.24) is of the form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix} = \begin{pmatrix} \lambda_i x_{i1} \\ \lambda_i x_{i2} \\ \vdots \\ \lambda_i x_{in} \end{pmatrix}. \tag{6.25}$$

Collectively they can be written as

$$\begin{aligned} &\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & \ldots & \lambda_n x_{n1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \ldots & \lambda_n x_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1 x_{1n} & \lambda_2 x_{2n} & \ldots & \lambda_n x_{nn} \end{pmatrix} \\ &\qquad = \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}. \end{aligned} \tag{6.26}$$

To simplify the writing, let

$$S = \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{pmatrix}, \tag{6.27}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \tag{6.28}$$

and write (6.26) as

$$AS = S\Lambda. \tag{6.29}$$

Multiplying both sides of this equation by S^{-1} from the left, we obtain

$$S^{-1}AS = \Lambda. \tag{6.30}$$

Thus, by using the matrix of eigenvectors and its inverse, it is possible to transform a matrix A to a diagonal matrix whose elements are the eigenvalues of A. The transformation expressed by (6.30) is referred to as the *diagonalization* of matrix A.

Example 6.4.1. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, find S such that $S^{-1}AS$ is a diagonal matrix. Show that the elements of $S^{-1}AS$ are the eigenvalues of A.

Solution 6.4.1. Since the secular equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3) = 0,$$

the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 3$. The eigenvectors are, respectively, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore

$$S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

It can be readily checked that $S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

$$S^{-1}AS = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Note that the diagonalizing matrix S is not necessary unitary. However, if the eigenvectors are orthogonal, then we can normalize the eigenvectors and form an orthonormal set. The matrix with members of this orthonormal set as columns is a unitary matrix. The diagonalization process becomes a unitary similarity transformation which is much more convenient and useful.

The two eigenvectors in the above example are orthogonal, since

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

Normalizing them, we get

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The matrix constructed with these two normalized eigenvectors is

$$U = \begin{pmatrix} \mathbf{u}_1 \; \mathbf{u}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

which is an orthogonal matrix. The transformation

$$\widetilde{U}AU = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

is a unitary similarity transformation.

First we have eliminated the step of finding the inverse of U, since U is an orthogonal matrix, the inverse of U is simply its transpose. More importantly, U represents a rotation as discuss in Sect. 6.3. If we rotate the two original coordinate axes to coincide with $\mathbf{u}_1$ and $\mathbf{u}_2$, then with respect to this rotated axes, A is diagonal.

The coordinate axes of a reference system, in which the matrix is diagonal, are known as *principal axes.* In this example, $\mathbf{u}_1$ and $\mathbf{u}_2$ are the unit vectors along the principal axes. From the components of $\mathbf{u}_1$, we can easily find the orientation of the principal axes. Let θ_1 be the angle between $\mathbf{u}_1$ and the original horizontal axis, then

$$\mathbf{u}_1 = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which gives $\theta_1 = -\pi/4$. This means that to get the principal axes, we have to rotate the original coordinate axes 45° clockwise. For consistence check, we can calculate θ_2, the angle between $\mathbf{u}_2$ and the original horizontal axis. Since

$$\mathbf{u}_2 = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$\theta_2 = +\pi/4$. Therefore the angle between θ_1 and θ_2 is $\pi/2$. This shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are perpendicular to each other, as they must.

Since $\theta_2 = \pi/2 + \theta_1$, $\cos\theta_2 = -\sin\theta_1$, and $\sin\theta_2 = \cos\theta_1$, the unitary matrix U can be written as

$$U = \begin{pmatrix} \mathbf{u}_1 \; \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix},$$

which, as seen in (6.17), is indeed a rotation matrix.

6.4.3 Quadratic Forms

A quadratic form is a homogeneous second degree expression in n variables. For example,

$$Q(x_1, x_2) = 5x_1^2 - 4x_1x_2 + 2x_2^2$$

is a quadratic form in x_1 and x_2. Through a change of variables, this expression can be transformed into a form in which there is no crossproduct term. Such a form is known as *canonical form*. Quadratic forms are important because they occur in a wide variety of applications.

The first step to change it into a canonical form is to separate the crossproduct term into two equal terms, $(4x_1x_2 = 2x_1x_2 + 2x_2x_1)$, so that $Q(x_1, x_2)$ can be written as

$$Q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{6.31}$$

where the coefficient matrix

$$C = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

is symmetric. As we shall see in Sect. 6.5 that symmetric matrices can always be diagonalized. In this particular case, we can first find the eigenvalues and eigenvectors of C.

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 6) = 0.$$

Corresponding to $\lambda_1 = 1$ and $\lambda_2 = 6$, the two normalized eigenvectors are found to be, respectively,

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Therefore the orthogonal matrix

$$U = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

will diagonalize the coefficient matrix

$$\widetilde{U}CU = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

If we make a change of variables

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = U \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and take the transpose of both sides

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \widetilde{U},$$

we can write (6.31) as

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix} \widetilde{U} C U \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1^2 + 6u_2^2, \tag{6.32}$$

which is in a canonical form, i.e., it has no crossterm.

Note that the transformation matrix T defined in (6.21) is equal to $\widetilde{U}$.

Example 6.4.2. Show that the following equation

$$9x^2 - 4xy + 6y^2 - 2\sqrt{5}x - 4\sqrt{6}y = 15$$

describes an ellipse by transforming it into a standard conic section form. Where is the center and what are the lengths of its major and minor axes?

Solution 6.4.2. The quadratic part of the equation can be written as

$$9x^2 - 4xy + 6y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are given by

$$\begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = (\lambda - 5)(\lambda - 10) = 0.$$

The normalized eigenvectors corresponding to $\lambda = 5$ and $\lambda = 10$ are found to be, respectively,

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Therefore the orthogonal matrix

$$U = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

diagonalizes the coefficient matrix

$$\widetilde{U} C U = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}.$$

Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = U \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

which is equivalent to

$$x = \frac{1}{\sqrt{5}}\left(x' - 2y'\right), \quad y = \frac{1}{\sqrt{5}}\left(2x' + y'\right),$$

then the equation can be written as

$$\left(x' \ y' \right) \widetilde{U}CU \begin{pmatrix} x' \\ y' \end{pmatrix} - 2\sqrt{5}\frac{1}{\sqrt{5}}\left(x' - 2y'\right) - 4\sqrt{5}\frac{1}{\sqrt{5}}\left(2x' + y'\right) = 15,$$

or

$$5x'^2 + 10y'^2 - 10x' = 15,$$
$$x'^2 + 2y'^2 - 2x' = 3.$$

Using $(x' - 1)^2 = x'^2 - 2x' + 1$, the last equation becomes

$$(x' - 1)^2 + 2y'^2 = 4,$$

or

$$\frac{(x' - 1)^2}{4} + \frac{y'^2}{2} = 1,$$

which is the standard form of an ellipse. The center of the ellipse is at $x = 1/\sqrt{5}$, $y = 2/\sqrt{5}$ (corresponding to $x' = 1$, $y' = 0$). The length of the major axis is $2\sqrt{4} = 4$, that of the minor axis is $2\sqrt{2}$.

To transform the equation into this standard form, we have rotated the coordinate axes. The major axis of the ellipse is along the vector $\mathbf{v}_1$ and the minor axis is along $\mathbf{v}_2$. Since

$$\mathbf{v}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

the major axis of the ellipse makes an angle θ with respect to the horizontal coordinate axis and $\theta = \cos^{-1}(1/\sqrt{5})$.

6.5 Hermitian Matrix and Symmetric Matrix

6.5.1 Definitions

Real Matrix

If $A^* = A$, then $a_{ij} = a_{ij}^*$. Since every element of this matrix is real, it is called a *real matrix*.

Imaginary Matrix

If $A^* = -A$, this implies that $a_{ij} = -a_{ij}^*$. Every element of this matrix is purely imaginary or zero, so it is called a *imaginary matrix*.

Hermitian Matrix

A square matrix is called hermitian if $A^\dagger = A$. It is easy to show that the elements of a hermitian matrix satisfy the relation $a_{ij}^* = a_{ji}$. Hermitian matrix is very important in quantum mechanics.

Symmetric Matrix

If the elements of the matrix are all real, a hermitian matrix is just a symmetric matrix. Symmetric matrix is of great importance in classical physics, hermitian matrix is essential in quantum mechanics.

Antihermitian Matrix and Antisymmetric Matrix

Finally, a matrix is called antihermitian or *skew-hermitian* if

$$A^\dagger = -A, \tag{6.33}$$

which implies $a_{ij}^* = -a_{ji}$.

Again, if the elements of the antihermitian matrix are all real, then the matrix is just an antisymmetric matrix.

6.5.2 Eigenvalues of Hermitian Matrix

- The eigenvalues of a hermitian (or real symmetric) matrix are all real.

Let A be a hermitian matrix, and $\mathbf{x}$ be the nontrivial eigenvector belonging to eigenvalue λ

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{6.34}$$

Take the hermitian conjugate of the equation

$$\mathbf{x}^\dagger A^\dagger = \lambda^*\mathbf{x}^\dagger. \tag{6.35}$$

Note that λ is only a number (real or complex), its hermitian conjugate is just the complex conjugate, it can be multiplied either from left or from the right.

Multiply (6.34) by $\mathbf{x}^\dagger$ from the left

$$\mathbf{x}^\dagger A\mathbf{x} = \lambda\mathbf{x}^\dagger\mathbf{x}.$$

Multiply (6.35) by $\mathbf{x}$ from the right

$$\mathbf{x}^\dagger A^\dagger\mathbf{x} = \lambda^*\mathbf{x}^\dagger\mathbf{x}.$$

Subtract it from the preceding equation

$$(\lambda - \lambda^*)\,\mathbf{x}^\dagger\mathbf{x} = \mathbf{x}^\dagger(A - A^\dagger)\mathbf{x}.$$

But A is hermitian, $A = A^\dagger$, so

$$(\lambda - \lambda^*)\, \mathbf{x}^\dagger \mathbf{x} = 0.$$

Since $\mathbf{x}^\dagger \mathbf{x} \neq 0$, it follows that $\lambda = \lambda^*$. That is, λ is real.

For real symmetric matrices, the proof is identical, since if the matrix is real, a hermitian matrix is a symmetric matrix.

- If two eigenvalues of a hermitian (or a real symmetric) matrix are different, the corresponding eigenvectors are orthogonal.

  Let

$$\begin{aligned} A\mathbf{x}_1 &= \lambda_1 \mathbf{x}_1, \\ A\mathbf{x}_2 &= \lambda_2 \mathbf{x}_2. \end{aligned}$$

Multiply the first equation by $\mathbf{x}_2^\dagger$ from the left

$$\mathbf{x}_2^\dagger A \mathbf{x}_1 = \lambda_1 \mathbf{x}_2^\dagger \mathbf{x}_1.$$

Take the hermitian conjugate of the second equation and multiply by $\mathbf{x}_1$ from the right

$$\mathbf{x}_2^\dagger A \mathbf{x}_1 = \lambda_2 \mathbf{x}_2^\dagger \mathbf{x}_1,$$

where we have used the facts that $(A\mathbf{x}_2)^\dagger = \mathbf{x}_2^\dagger A^\dagger$, $A^\dagger = A$, and $\lambda_2 = \lambda_2^*$. Subtracting these two equations, we have

$$(\lambda_1 - \lambda_2)\mathbf{x}_2^\dagger \mathbf{x}_1 = 0.$$

Since $\lambda_1 \neq \lambda_2$, it follows:

$$\mathbf{x}_2^\dagger \mathbf{x}_1 = 0.$$

Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal. For real symmetric matrices, the proof is the same.

6.5.3 Diagonalizing a Hermitian Matrix

- A hermitian (or a real symmetric) matrix can be diagonalized by a unitary (or a real orthogonal) matrix.

If the eigenvalues of the matrix are all distinct, the matrix can be diagonalized by a similarity transformation as we discussed before. Here we only need to show that even if the eigenvalues are degenerate, as long as the matrix is hermitian, it can always be diagonalized. We will prove it by actually constructing a unitary matrix that will diagonalize a degenerate hermitian matrix.

Let λ_1 be a repeated eigenvalue of the $n \times n$ hermitian matrix H, let $\mathbf{x}_1$ be a normalized eigenvector corresponding to λ_1. We can take any n linearly independent vectors with the only condition that the first one is $\mathbf{x}_1$ and construct with the Gram–Schmidt process an orthonormal set of n vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, each of them has n elements.

Let U_1 be the matrix with $\mathbf{x}_i$ as its ith column

$$U_1 = \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{pmatrix},$$

as we have shown this automatically makes U_1 an unitary matrix. The unitary transformation $U_1^\dagger H U_1$ has exactly the same set of eigenvalues as H, since they have the same characteristic polynomial

$$\begin{aligned} \left| U_1^\dagger H U_1 - \lambda I \right| &= \left| U_1^{-1} H U_1 - \lambda U_1^{-1} U_1 \right| = \left| U_1^{-1} (H - \lambda I) U_1 \right| \\ &= \left| U_1^{-1} \right| |H - \lambda I| \, |U_1| = |H - \lambda I| \,. \end{aligned}$$

Furthermore, since H is hermitian, $U_1^\dagger H U_1$ is also hermitian, since

$$\left(U_1^\dagger H U_1 \right)^\dagger = (H U_1)^\dagger \left(U_1^\dagger \right)^\dagger = U_1^\dagger H^\dagger U_1 = U_1^\dagger H U_1.$$

Now

$$U_1^\dagger H U_1 = \begin{pmatrix} x_{11}^* & x_{12}^* & \ldots & x_{1n}^* \\ x_{21}^* & x_{22}^* & \ldots & x_{2n}^* \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}^* & x_{n2}^* & \ldots & x_{nn}^* \end{pmatrix} H \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} x_{11}^* & x_{12}^* & \ldots & x_{1n}^* \\ x_{21}^* & x_{22}^* & \ldots & x_{2n}^* \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}^* & x_{n2}^* & \ldots & x_{nn}^* \end{pmatrix} \begin{pmatrix} \lambda_1 x_{11} & h_{21} & \ldots & h_{n1} \\ \lambda_1 x_{12} & h_{22} & \ldots & h_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1 x_{1n} & h_{2n} & \ldots & h_{nn} \end{pmatrix},$$

where we have used the fact that $\mathbf{x}_1$ is an eigenvector of H belonging to the eigenvalue λ_1

$$H \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix} = \lambda_1 \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix},$$

and have written

$$H \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix} = \begin{pmatrix} h_{i1} \\ h_{i2} \\ \vdots \\ h_{in} \end{pmatrix}$$

for $i \neq 1$. Furthermore

$$U_1^{\dagger} H U_1 = \begin{pmatrix} x_{11}^* & x_{12}^* & \dots & x_{1n}^* \\ x_{21}^* & x_{22}^* & \dots & x_{2n}^* \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}^* & x_{n2}^* & \dots & x_{nn}^* \end{pmatrix} \begin{pmatrix} \lambda_1 x_{11} & h_{21} & \dots & h_{n1} \\ \lambda_1 x_{12} & h_{22} & \dots & h_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1 x_{1n} & h_{2n} & \dots & h_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \alpha_{22} & \alpha_{32} & \dots & \alpha_{n2} \\ 0 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} \end{pmatrix}.$$

The first column is determined by the orthonormal condition

$$\begin{pmatrix} x_{i1}^* & x_{i2}^* & \cdots & x_{in}^* \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix} = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

The first row must be the transpose of the first column because $U_1^{\dagger} H U_1$ is hermitian (or real symmetric) and λ_1 is real and the complex conjugate of zero is itself. The crucial fact in this process is that the last $n-1$ elements of the first row are all zero. This is what distinguishes hermitian (or real symmetric) matrices from other square matrices.

If λ_1 is a twofold degenerate eigenvalue of H, then in the characteristic polynomial $p(\lambda) = |H - \lambda I|$, there must be a factor $(\lambda_1 - \lambda)^2$. Since

$$\begin{aligned} p(\lambda) &= |H - \lambda I| = \left| U_1^{\dagger} H U_1 - \lambda I \right| \\ &= \begin{vmatrix} \lambda_1 - \lambda & 0 & 0 & \dots & 0 \\ 0 & \alpha_{22} - \lambda & \alpha_{32} & \dots & \alpha_{n2} \\ 0 & \alpha_{23} & \alpha_{33} - \lambda & \dots & \alpha_{n3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} - \lambda \end{vmatrix} \\ &= (\lambda_1 - \lambda) \begin{vmatrix} \alpha_{22} - \lambda & \alpha_{32} & \dots & \alpha_{n2} \\ \alpha_{23} & \alpha_{33} - \lambda & \dots & \alpha_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} - \lambda \end{vmatrix}, \end{aligned}$$

the part

$$\begin{vmatrix} \alpha_{22} - \lambda & \alpha_{32} & \dots & \alpha_{n2} \\ \alpha_{23} & \alpha_{33} - \lambda & \dots & \alpha_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} - \lambda \end{vmatrix}$$

must contain another factor of $(\lambda_1 - \lambda)$. In other words, if we define H_1 as the $(n-1) \times (n-1)$ submatrix

$$\begin{pmatrix} \alpha_{22} & \alpha_{32} & \ldots & \alpha_{n2} \\ \alpha_{23} & \alpha_{33} & \ldots & \alpha_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{2n} & \alpha_{3n} & \ldots & \alpha_{nn} \end{pmatrix} = H_1,$$

then λ_1 must be an eigenvalue of H_1. Thus we can repeat the process and construct an orthonormal set of $n-1$ column vectors with the first one being the eigenvector of H_1 belonging to the eigenvalue λ_1. Let this orthonormal set be

$$\mathbf{y}_1 = \begin{pmatrix} y_{22} \\ y_{23} \\ \vdots \\ y_{2n} \end{pmatrix}, \mathbf{y}_2 = \begin{pmatrix} y_{32} \\ y_{33} \\ \vdots \\ y_{3n} \end{pmatrix}, \ldots, \mathbf{y}_{n-1} = \begin{pmatrix} y_{n2} \\ y_{n3} \\ \vdots \\ y_{nn} \end{pmatrix},$$

and let U_2 be another unitary matrix defined as

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & y_{22} & y_{32} & \cdots & y_{n2} \\ 0 & y_{23} & y_{33} & \cdots & y_{n3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & y_{2n} & y_{3n} & \cdots & y_{nn} \end{pmatrix},$$

then the unitary transformation $U_2^\dagger(U_1^\dagger HU_1)U_2$ can be shown as

$$U_2^\dagger(U_1^\dagger HU_1)U_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & y_{22}^* & y_{23}^* & \cdots & y_{2n}^* \\ 0 & y_{32}^* & y_{33}^* & \cdots & y_{3n}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & y_{2n}^* & y_{3n}^* & \cdots & y_{nn}^* \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \alpha_{22} & \alpha_{32} & \ldots & \alpha_{n2} \\ 0 & \alpha_{23} & \alpha_{33} & \ldots & \alpha_{n3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{2n} & \alpha_{3n} & \ldots & \alpha_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & y_{22} & y_{32} & \cdots & y_{n2} \\ 0 & y_{23} & y_{33} & \cdots & y_{n3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & y_{2n} & y_{3n} & \cdots & y_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_1 & 0 & \ldots & 0 \\ 0 & 0 & \beta_{33} & \ldots & \beta_{n3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \beta_{3n} & \ldots & \beta_{nn} \end{pmatrix}.$$

If λ_1 is m-fold degenerate, we repeat this process m times. The rest can be diagonalized by the eigenvectors belonging to different eigenvalues. After the $n \times n$ matrix is so transformed $n-1$ times, it will become diagonal.

Let us define

$$U = U_1 U_2 \cdots U_{n-1},$$

then U is unitary because all U_i are unitary. Consequently the hermitian matrix H is diagonalized by the unitary transformation $U^\dagger HU$ and the theorem is established.

This construction leads to the following important corollary.

- Every $n \times n$ hermitian (or real symmetric) matrix has n orthogonal eigenvectors regardless of the degeneracy of its eigenvalues.

This is because $U^{\dagger}HU = \Lambda$, where the elements of the diagonal matrix Λ are the eigenvalues of H. Since $U^{\dagger} = U^{-1}$, it follows from the equation $U(U^{\dagger}HU) = U\Lambda$ that $HU = U\Lambda$, which shows that each column of U is an normalized eigenvector of H.

The following example illustrates this process.

Example 6.5.1. Find an unitary matrix that will diagonalize the hermitian matrix

$$H = \begin{pmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & \mathrm{i} \\ 1 & -\mathrm{i} & 2 \end{pmatrix}.$$

Solution 6.5.1. The eigenvalues of H are the roots of the characteristic equation

$$p(\lambda) = \begin{vmatrix} 2-\lambda & \mathrm{i} & 1 \\ -\mathrm{i} & 2-\lambda & \mathrm{i} \\ 1 & -\mathrm{i} & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 9\lambda = -\lambda(\lambda-3)^2 = 0.$$

Therefore the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are

$$\lambda_1 = 3, \quad \lambda_2 = 3, \quad \lambda_3 = 0.$$

It is seen that $\lambda = \lambda_1 = \lambda_2 = 3$ is twofold degenerate. Let one of the eigenvectors corresponding to λ_1 be

$$\mathbf{E}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

so

$$\begin{pmatrix} 2-\lambda_1 & \mathrm{i} & 1 \\ -\mathrm{i} & 2-\lambda_1 & \mathrm{i} \\ 1 & -\mathrm{i} & 2-\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & \mathrm{i} & 1 \\ -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -\mathrm{i} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The three equations

$$\begin{aligned} -x_1 + \mathrm{i}x_2 + x_3 &= 0, \\ -\mathrm{i}x_1 - x_2 + \mathrm{i}x_3 &= 0, \\ x_1 - \mathrm{i}x_2 - x_3 &= 0 \end{aligned}$$

are identical to each other. For example, multiply the middle one by i will change it to the last one. The equation

$$x_1 - \mathrm{i}x_2 - x_3 = 0 \tag{6.36}$$

has an infinite number of solutions. A simple choice is to set $x_2 = 0$, then $x_1 = x_3$. Therefore

$$\mathbf{E}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is an eigenvector. Certainly

$$\mathbf{E}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{E}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

are linearly independent. Now let us use the Gram–Schmidt process to find an orthonormal set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

$$\mathbf{x}_1 = \frac{\mathbf{E}_1}{\|\mathbf{E}_1\|} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$\mathbf{E}_2$ is already normalized and it is orthogonal to $\mathbf{E}_1$, so

$$\mathbf{x}_2 = \mathbf{E}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{aligned} \mathbf{x}_3' &= \mathbf{E}_3 - (\mathbf{E}_3, \mathbf{x}_1)\,\mathbf{x}_1 - (\mathbf{E}_3, \mathbf{x}_2)\,\mathbf{x}_2 \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \end{aligned}$$

$$\mathbf{x}_3 = \frac{\mathbf{x}_3'}{\|\mathbf{x}_3'\|} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Form a unitary matrix with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$

$$U_1 = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

The unitary similarity transformation of H by U_1 is

$$U_1^\dagger H U_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & \mathrm{i} \\ 1 & -\mathrm{i} & 2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -\sqrt{2}\mathrm{i} \\ 0 & \sqrt{2}\mathrm{i} & 1 \end{pmatrix}.$$

Since H and $U_1^\dagger H U_1$ have the same set of eigenvalues, therefore $\lambda = 3$ and $\lambda = 0$ must be the eigenvalue of the submatrix

$$H_1 = \begin{pmatrix} 2 & -\sqrt{2}\mathrm{i} \\ \sqrt{2}\mathrm{i} & 1 \end{pmatrix}.$$

This can also be shown directly. The two normalized eigenvector of H_1 corresponding to $\lambda = 3$ and $\lambda = 0$ are found, respectively, to be

$$\mathbf{y}_1 = \begin{pmatrix} -\frac{\sqrt{6}}{3}\mathrm{i} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} \frac{\sqrt{3}}{3}\mathrm{i} \\ \frac{\sqrt{6}}{3} \end{pmatrix}.$$

Therefore

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{3}\mathrm{i} & \frac{\sqrt{3}}{3}\mathrm{i} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{pmatrix},$$

and

$$U = U_1 U_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{3}\mathrm{i} & \frac{\sqrt{3}}{3}\mathrm{i} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{6}}{3}\mathrm{i} & \frac{\sqrt{3}}{3}\mathrm{i} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}.$$

It can be easily checked that

$$U^\dagger H U = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3}\mathrm{i} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3}\mathrm{i} & -\frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & \mathrm{i} \\ 1 & -\mathrm{i} & 2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{6}}{3}\mathrm{i} & \frac{\sqrt{3}}{3}\mathrm{i} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is indeed diagonal and the diagonal elements are the eigenvalues. Furthermore, the three columns of U are indeed three orthogonal eigenvectors of H

$$H\mathbf{u}_1 = \lambda_1\mathbf{u}_1 : \quad \begin{pmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & \mathrm{i} \\ 1 & -\mathrm{i} & 2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} = 3 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix},$$

$$H\mathbf{u}_2 = \lambda_2\mathbf{u}_2 : \quad \begin{pmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & \mathrm{i} \\ 1 & -\mathrm{i} & 2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3}\mathrm{i} \\ -\frac{\sqrt{6}}{6} \end{pmatrix} = 3 \begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3}\mathrm{i} \\ -\frac{\sqrt{6}}{6} \end{pmatrix},$$

$$H\mathbf{u}_3 = \lambda_3\mathbf{u}_3 : \quad \begin{pmatrix} 2 & \mathrm{i} & 1 \\ -\mathrm{i} & 2 & \mathrm{i} \\ 1 & -\mathrm{i} & 2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3}\mathrm{i} \\ -\frac{\sqrt{3}}{3} \end{pmatrix} = 0 \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3}\mathrm{i} \\ -\frac{\sqrt{3}}{3} \end{pmatrix}.$$

We have followed the steps of the proof in order to illustrate the procedure. Once it is established, we can make use of the theorem and the process of finding the eigenvectors can be simplified considerably.

In this example, one can find the eigenvector for the nondegenerate eigenvalue the usual way. For the degenerate eigenvalue $\lambda = 3$, the components (x_1, x_2, x_3) of the corresponding eigenvectors must satisfy

$$x_1 - \mathrm{i}x_2 - x_3 = 0,$$

as shown in (6.36). This equation can be written as $x_2 = \mathrm{i}(x_3 - x_1)$. So in general

$$\mathbf{u} = \begin{pmatrix} x_1 \\ \mathrm{i}(x_3 - x_1) \\ x_3 \end{pmatrix},$$

where x_1 and x_3 are arbitrary. It is seen that $\mathbf{u}_1$ is just the normalized eigenvector obtained by choosing $x_1 = x_3$

$$\mathbf{u}_1 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The other eigenvector must also satisfy the same equation and be orthogonal to $\mathbf{u}_1$. Thus

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \mathrm{i}(x_3 - x_1) \\ x_3 \end{pmatrix} = 0,$$

which gives $x_1 + x_3 = 0$, or $x_3 = -x_1$. Normalizing the vector $\begin{pmatrix} x_1 \\ -2x_1 \\ -x_1 \end{pmatrix}$, one obtains the other eigenvector belonging to $\lambda = 3$

$$\mathbf{u}_2 = \frac{\sqrt{6}}{6} \begin{pmatrix} 1 \\ -2\mathrm{i} \\ -1 \end{pmatrix}.$$

6.5.4 Simultaneous Diagonalization

If A and B are two hermitian matrices of the same order, the following important question often arises. Can they be simultaneously diagonalized by the same matrix S? That is to say, does there exist a basis in which they are both diagonal? The answer is yes if and only if they commute.

First we will show that if they can be simultaneously diagonalized, then they must commute. That is, if

$$D_1 = S^{-1}AS \quad \text{and} \quad D_2 = S^{-1}BS,$$

where D_1 and D_2 are both diagonal matrices, then $AB = BA$.

This follows from the fact

$$D_1D_2 = S^{-1}ASS^{-1}BS = S^{-1}ABS,$$
$$D_2D_1 = S^{-1}BSS^{-1}AS = S^{-1}BAS.$$

Since diagonal matrices of the same order always commute ($D_1D_2 = D_2D_1$), so we have

$$S^{-1}ABS = S^{-1}BAS.$$

Multiplying S from the left and S^{-1} from the right, we obtain $AB = BA$.

Now we will show that the converse is also true. That is, if they commute, then they can be simultaneously diagonalized. First let A and B be 2×2 matrices. Since hermitian matrix is always diagonalizable, let S be the unitary matrix that diagonalizes A

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues of A. Let

$$S^{-1}BS = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Now

$$S^{-1}ABS = S^{-1}ASS^{-1}BS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}\lambda_1 & b_{12}\lambda_1 \\ b_{21}\lambda_2 & b_{22}\lambda_2 \end{pmatrix},$$
$$S^{-1}BAS = S^{-1}BSS^{-1}AS = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} b_{11}\lambda_1 & b_{12}\lambda_2 \\ b_{21}\lambda_1 & b_{22}\lambda_2 \end{pmatrix}.$$

Since $AB = BA$, so $S^{-1}ABS = S^{-1}BAS$

$$\begin{pmatrix} b_{11}\lambda_1 & b_{12}\lambda_1 \\ b_{21}\lambda_2 & b_{22}\lambda_2 \end{pmatrix} = \begin{pmatrix} b_{11}\lambda_1 & b_{12}\lambda_2 \\ b_{21}\lambda_1 & b_{22}\lambda_2 \end{pmatrix}.$$

It follows that:

$$b_{21}\lambda_2 = b_{21}\lambda_1, \quad b_{12}\lambda_1 = b_{12}\lambda_2.$$

If $\lambda_2 \neq \lambda_1$, then $b_{12} = 0$ and $b_{21} = 0$. In other words

$$S^{-1}BS = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}.$$

Therefore A and B are simultaneously diagonalized.

If $\lambda_2 = \lambda_1 = \lambda$, we cannot conclude that $S^{-1}BS$ is diagonal. However, in this case

$$S^{-1}AS = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Moveover, since B is hermitian, so the unitary similarity transform $S^{-1}BS$ is also hermitian, therefore $S^{-1}BS$ is diagonalizable. Let T be the unitary matrix that diagonalize $S^{-1}BS$

$$T^{-1}\left(S^{-1}BS\right)T = \begin{pmatrix} \lambda_1' & 0 \\ 0 & \lambda_2' \end{pmatrix}.$$

On the other hand,

$$T^{-1}\left(S^{-1}AS\right)T = T^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}T = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}T^{-1}T = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Thus the product matrix $U = ST$ diagonalizes both A and B. Therefore, with or without degeneracy, as long as A and B commute, they can be simultaneously diagonalized.

Although we have used 2×2 matrices for illustration, the same "proof" can obviously be applied to matrices of higher order.

Example 6.5.2. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Can A and B be simultaneously diagonalized? If so, find the unitary matrix that diagonalized them.

Solution 6.5.2.

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 7 & 8 \end{pmatrix},$$

$$BA = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 7 & 8 \end{pmatrix}.$$

Thus $[A, B] = 0$, therefore they can be simultaneously diagonalized

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 3) = 0.$$

The normalized eigenvectors corresponding to $\lambda = 1, 3$ are, respectively,

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad S^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$S^{-1}AS = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$
$$S^{-1}BS = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Thus they are simultaneously diagonalized. It also shows that 1 and 5 are the eigenvalues of B. This can be easily verified, since

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 5) = 0.$$

If we diagonalize B first, we will get exactly the same result.

6.6 Normal Matrix

A square matrix is said to be normal if and only if it commutes with its hermitian conjugate. That is, A is normal, if and only if

$$AA^{\dagger} = A^{\dagger}A. \tag{6.37}$$

It can be easily shown that all hermitian (or real symmetric), antihermitian (or real antisymmetric), and unitary (or real orthogonal) matrices are normal. All we have to do is to substitute these matrices into (6.37). By virtue of their definition, it is immediately clear that the two sides of the equation are indeed the same.

So far we have shown that every hermitian matrix is diagonalizable by a unitary similarity transformation. In what follows, we will prove the generalization of this theorem that every normal matrix is diagonalizable.

First, if the square matrix A is given, that means all elements of A are known, so we can take its hermitian conjugate $A^{\dagger}$. Then let

$$B = \frac{1}{2}(A + A^\dagger),$$
$$C = \frac{1}{2\mathrm{i}}(A - A^\dagger).$$

So

$$A = B + \mathrm{i}C. \tag{6.38}$$

Since $(A^\dagger)^\dagger = A$ and $(A+B)^\dagger = A^\dagger + B^\dagger$,

$$B^\dagger = \frac{1}{2}(A + A^\dagger)^\dagger = \frac{1}{2}(A^\dagger + A) = B,$$
$$C^\dagger = \frac{1}{2\mathrm{i}^*}(A - A^\dagger)^\dagger = -\frac{1}{2\mathrm{i}}(A^\dagger - A) = C.$$

Thus, B and C are both hermitian. In other words, a square matrix can always be decomposed into two hermitian matrices as shown in (6.38). Furthermore

$$BC = \frac{1}{4\mathrm{i}}\left(A^2 - AA^\dagger + A^\dagger A - A^{\dagger 2}\right),$$
$$CB = \frac{1}{4\mathrm{i}}\left(A^2 - A^\dagger A + AA^\dagger - A^{\dagger 2}\right).$$

It is clear that if $A^\dagger A = AA^\dagger$, then $BC = CB$. In other words, if A is normal, then B and C commute.

We have shown in Sect. 6.5 that if B and C commute, then they can be simultaneously diagonalized. That is, we can find a unitary matrix S such that $S^{-1}BS$ and $S^{-1}CS$ are both diagonal. Since

$$S^{-1}AS = S^{-1}BS + \mathrm{i}S^{-1}CS,$$

it follows that $S^{-1}AS$ must also be diagonal.

Conversely, if $S^{-1}AS = D$ is diagonal, then

$$\left(S^{-1}AS\right)^\dagger = S^{-1}A^\dagger S = D^\dagger = D^*,$$

since S is unitary and D is diagonal. It follows that:

$$S^{-1}AA^\dagger S = (S^{-1}AS)\left(S^{-1}A^\dagger S\right) = DD^*,$$
$$S^{-1}A^\dagger AS = (S^{-1}A^\dagger S)(S^{-1}AS) = D^*D.$$

Since $DD^* = D^*D$, clearly $AA^\dagger = A^\dagger A$. Therefore we conclude.

- A matrix can be diagonalized by a unitary similarity transformation if and only if it is normal.

Thus both hermitian and unitary matrices are diagonalizable by a unitary similarity transformation.

The eigenvalues of a hermitian matrix are always real. This is the reason why in quantum mechanics observable physical quantities are associated with the eigenvalues of hermitian operators, because the result of any measurement is, of course, a real number. However, the eigenvectors of a hermitian matrix may be complex, therefore the unitary matrix that diagonalizes the hermitian matrix is, in general, complex.

A real symmetric matrix is a hermitian matrix, therefore its eigenvalues must also be real. Since the matrix and the eigenvalues are both real, the eigenvectors can be taken to be real. Therefore, the diagonalizing matrix is a real orthogonal matrix.

Unitary matrices, including real orthogonal matrices, can be diagonalized by a unitary similarity transformation. However, the eigenvalues and eigenvectors of a unitary matrix are, in general, complex. Therefore the diagonalizing matrix is not a real orthogonal matrix, but a complex unitary matrix. For example, the rotation matrix is a real orthogonal matrix, but, in general, it can only be diagonalized by a complex unitary matrix.

6.7 Functions of a Matrix

6.7.1 Polynomial Functions of a Matrix

Any square matrix A can be multiplied by itself. The associative law of matrix multiplication guarantees that the operation of A times itself n times, which is denoted as A^n, is an unambiguous operation. Thus

$$A^m A^n = A^{m+n}.$$

Moreover, we have defined the inverse A^{-1} of a nonsingular matrix in such a way that $AA^{-1} = A^{-1}A = I$. Therefore it is natural to define

$$A^0 = A^{1-1} = AA^{-1} = I, \quad \text{and} \quad A^{-n} = \left(A^{-1}\right)^n.$$

With these definitions, we can define polynomial functions of a square matrix in exactly the same way as scalar polynomials.

For example, if $f(x) = x^2 + 5x + 4$, and $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, we define $f(A)$, as

$$f(A) = A^2 + 5A + 4.$$

Since

$$A^2 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix},$$

$$f(A) = \begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix} + 5\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 18 & 30 \end{pmatrix}.$$

It is interesting to note that $f(A)$ can also be evaluated by using the factored terms of $f(x)$. For example

$$f(x) = x^2 + 5x + 4 = (x+1)(x+4),$$

so

$$\begin{aligned} f(A) &= (A+I)(A+4I) \\ &= \left[\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 18 & 30 \end{pmatrix}. \end{aligned}$$

Example 6.7.1. Find $f(A)$, if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad f(x) = \frac{x}{x^2-1}.$$

Solution 6.7.1.

$$f(A) = \frac{A}{A^2 - I} = A\left(A^2 - I\right)^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 8 & 10 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Note that $f(A)$ can also be evaluated by partial fraction. Since

$$f(x) = \frac{x}{x^2-1} = \frac{1}{2}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1},$$

$$\begin{aligned} f(A) &= \frac{1}{2}(A-I)^{-1} + \frac{1}{2}(A+I)^{-1} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}^{-1} + \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}. \end{aligned}$$

6.7.2 Evaluating Matrix Functions by Diagonalization

When the square matrix A is similar to a diagonal matrix, the evaluation of $f(A)$ can be considerably simplified.

If A is diagonalizable, then

$$S^{-1}AS = D,$$

where D is a diagonal matrix. It follows that:

$$\begin{aligned} D^2 &= S^{-1}ASS^{-1}AS = S^{-1}A^2S, \\ D^k &= S^{-1}A^{k-1}SS^{-1}AS = S^{-1}A^kS. \end{aligned}$$

Thus

$$
\begin{aligned}
A^k &= SD^kS^{-1}, \\
A^n + A^m &= SD^nS^{-1} + SD^mS^{-1} = S\left(D^n + D^m\right)S^{-1}.
\end{aligned}
$$

If $f(A)$ is a polynomial, then

$$
f(A) = Sf(D)\,S^{-1}.
$$

Furthermore, since D is diagonal and the elements of D are the eigenvalues of A,

$$
D^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \lambda_n^k \end{pmatrix},
$$

$$
f(D) = \begin{pmatrix} f(\lambda_1) & 0 & \dots & 0 \\ 0 & f(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & f(\lambda_n) \end{pmatrix}.
$$

Therefore

$$
f(A) = S\begin{pmatrix} f(\lambda_1) & 0 & \dots & 0 \\ 0 & f(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & f(\lambda_n) \end{pmatrix}S^{-1}.
$$

Example 6.7.2. Find $f(A)$, if

$$
A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad f(x) = x^4 - 4x^3 + 6x^2 - x - 3.
$$

Solution 6.7.2. First find the eigenvalues and eigenvectors of A

$$
\begin{vmatrix} 0-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0.
$$

The eigenvectors corresponding to $\lambda_1 = 1$ and $\lambda_2 = 2$ are, respectively,

$$
\mathbf{u}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
$$

Therefore

$$
S = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix},
$$

and

$$D = S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Thus

$$f(A) = Sf(D)S^{-1} = S\begin{pmatrix} f(1) & 0 \\ 0 & f(2) \end{pmatrix} S^{-1}.$$

Since

$$f(1) = -1, \quad f(2) = 3,$$

$$f(A) = Sf(D)S^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -5 & -8 \\ 4 & 7 \end{pmatrix}.$$

Example 6.7.3. Find the matrix A such that

$$A^2 - 4A + 4I = \begin{pmatrix} 4 & 3 \\ 5 & 6 \end{pmatrix}.$$

Solution 6.7.3. Let us first diagonalize the right-hand side

$$\begin{vmatrix} 4-\lambda & 3 \\ 5 & 6-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 9) = 0.$$

The eigenvectors corresponding to $\lambda_1 = 1$ and $\lambda_2 = 9$ are, found to be, respectively,

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Thus

$$S = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}, \quad S^{-1} = \frac{1}{8}\begin{pmatrix} 5 & -3 \\ 1 & 1 \end{pmatrix},$$

and

$$D = S^{-1}\begin{pmatrix} 4 & 3 \\ 5 & 6 \end{pmatrix} S = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}.$$

Therefore

$$S^{-1}\left(A^2 - 4A + 4I\right)S = S^{-1}\begin{pmatrix} 4 & 3 \\ 5 & 6 \end{pmatrix} S = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}.$$

The left-hand side must also be diagonal, since the right-hand side is diagonal. Since we have shown that, as long as $S^{-1}AS$ is diagonal, $S^{-1}A^kS$ will be diagonal, we can assume

$$S^{-1}AS = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}.$$

It follows that:

$$S^{-1}\left(A^2-4A+4I\right)S=\begin{pmatrix} x_1^2-4x_1+4 & 0 \\ 0 & x_2^2-4x_2+4 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix},$$

which gives

$$\begin{aligned} x_1^2-4x_1+4&=1, \\ x_2^2-4x_2+4&=9. \end{aligned}$$

From the first equation we get $x_1=1,3$, and from the second equation we obtain $x_2=5,-1$. Therefore there are four possible combinations for $\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$, namely

$$\Lambda_1=\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \quad \Lambda_2=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Lambda_3=\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}, \quad \Lambda_4=\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus the original equation has four solutions

$$A_1=S\Lambda_1S^{-1}=\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\frac{1}{8}\begin{pmatrix} 5 & -3 \\ 1 & 1 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 5 & 3 \\ 5 & 7 \end{pmatrix},$$

and similarly

$$A_2=\frac{1}{4}\begin{pmatrix} 1 & -3 \\ -5 & -1 \end{pmatrix}, \quad A_3=\frac{1}{4}\begin{pmatrix} 15 & 3 \\ 5 & 17 \end{pmatrix}, \quad A_4=\frac{1}{2}\begin{pmatrix} 3 & -3 \\ -5 & 1 \end{pmatrix}.$$

For every scalar function that can be expressed as an infinite series, a corresponding matrix function can be defined. For example, with

$$\mathrm{e}^x=1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\cdots,$$

we can define

$$\mathrm{e}^A=I+A+\frac{1}{2}A^2+\frac{1}{3!}A^3+\cdots.$$

If A is diagonalizable, then

$$\begin{aligned} &S^{-1}AS=D, \quad A^n=SD^nS^{-1}, \\ &\mathrm{e}^A=S\left(I+D+\frac{1}{2}D^2+\frac{1}{3!}D^3+\cdots\right)S^{-1}, \end{aligned}$$

where

$$D=\begin{pmatrix} \lambda_1 & \ldots & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \ldots & \ldots & \ldots & \lambda_n \end{pmatrix}.$$

It follows that:

$$e^A = S \begin{pmatrix} 1+\lambda_1+\frac{1}{2}\lambda_1^2\cdots & \dots & \dots & 0 \\ 0 & 1+\lambda_2+\frac{1}{2}\lambda_2^2\cdots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1+\lambda_n+\frac{1}{2}\lambda_n^2\cdots \end{pmatrix} S^{-1}$$

$$= S \begin{pmatrix} e^{\lambda_1} & \dots & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n} \end{pmatrix} S^{-1}.$$

Example 6.7.4. Evaluate e^A if $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$.

Solution 6.7.4. Since A is symmetric, it is diagonalizable.

$$\begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 24 = 0,$$

which gives $\lambda = 6, -4$. The corresponding eigenvectors are found to be

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore

$$e^A = S \begin{pmatrix} e^6 & 0 \\ 0 & e^{-4} \end{pmatrix} S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^6 & 0 \\ 0 & e^{-4} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (e^6+e^{-4}) & (e^6-e^{-4}) \\ (e^6-e^{-4}) & (e^6+e^{-4}) \end{pmatrix}.$$

6.7.3 The Cayley–Hamilton Theorem

The famous Cayley–Hamilton theorem states that every square matrix satisfies its own characteristic equation.

That is, if $P(\lambda)$ is the characteristic polynomial of the nth order matrix A

$$P(\lambda) = |A - \lambda I| = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0,$$

then

$$P(A) = c_n A^n + c_{n-1} A^{n-1} + \cdots + c_0 I = 0.$$

To prove this theorem, let $\mathbf{x}_i$ be the eigenvector corresponding to the eigenvalue λ_i. So

$$P(\lambda_i) = 0, \quad A\mathbf{x}_i = \lambda_i \mathbf{x}_i.$$

Now

$$\begin{aligned} P(A)\mathbf{x}_i &= \left(c_n A^n + c_{n-1} A^{n-1} + \cdots + c_0 I\right) \mathbf{x}_i \\ &= \left(c_n \lambda_i^n + c_{n-1} \lambda_i^{n-1} + \cdots + c_0\right) \mathbf{x}_i \\ &= P(\lambda_i)\mathbf{x}_i = 0\mathbf{x}_i. \end{aligned}$$

Since this is true for any eigenvector of A, $P(A)$ must be a zero matrix.

For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3.$$

$$\begin{aligned} P(A) &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5-3-2 & 4-4 \\ 4-4 & 5-3-2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Inverse by Cayley–Hamilton Theorem

The Cayley–Hamilton theorem can be used to find the inverse of a square matrix. Starting with the characteristic equation of A

$$P(\lambda) = |A - \lambda I| = c_n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1 \lambda + c_0 = 0,$$

we have

$$P(A) = c_n A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I = 0.$$

Multiplying this equation from the left by A^{-1}, we obtain

$$A^{-1}P(A) = c_n A^{n-1} + c_{n-1} A^{n-2} + \cdots + c_1 I + c_0 A^{-1} = 0.$$

Thus

$$A^{-1} = -\frac{1}{c_0}\left(c_n A^{n-1} + c_{n-1} A^{n-2} + \cdots + c_1 I\right).$$

Example 6.7.5. Find A^{-1} by Cayley–Hamilton theorem if

$$A = \begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix}.$$

Solution 6.7.5.

$$P(\lambda) = \begin{pmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{pmatrix} = 6 - 11\lambda + 6\lambda^2 - \lambda^3,$$

$$\begin{aligned} P(A) &= 6I - 11A + 6A^2 - A^3 = 0, \\ A^{-1}P(A) &= 6A^{-1} - 11I + 6A - A^2 = 0, \\ A^{-1} &= \frac{1}{6}\left(A^2 - 6A + 11I\right), \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{6}\left[\begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix}\begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix} - 6\begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix} + 11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \\ &= \frac{1}{6}\left[\begin{pmatrix} 15 & 23 & -17 \\ -2 & 8 & -1 \\ 4 & 22 & -9 \end{pmatrix} - \begin{pmatrix} 30 & 42 & -30 \\ 0 & 24 & -6 \\ 12 & 48 & -18 \end{pmatrix} + \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}\right] \\ &= \frac{1}{6}\begin{pmatrix} -4 & -19 & 13 \\ -2 & -5 & 5 \\ -8 & -26 & 20 \end{pmatrix}. \end{aligned}$$

It can be readily verified that

$$A^{-1}A = \frac{1}{6}\begin{pmatrix} -4 & -19 & 13 \\ -2 & -5 & 5 \\ -8 & -26 & 20 \end{pmatrix}\begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

High Powers of a Matrix

An important application of the Cayley–Hamilton theorem is in the representation of high powers of a matrix. From the equation $P(A) = 0$, we have

$$A^n = -\frac{1}{c_n}\left(c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots + c_1 A + c_0 I\right). \tag{6.39}$$

Multiplying through by A

$$A^{n+1} = -\frac{1}{c_n}(c_{n-1}A^n + c_{n-2}A^{n-1} + \cdots + c_1 A^2 + c_0 A), \tag{6.40}$$

and substituting A^n from (6.39) into (6.40), we obtain

$$A^{n+1} = \left(\frac{c_{n-1}^2}{c_n^2} - \frac{c_{n-2}}{c_n}\right) A^{n-1} + \cdots + \left(\frac{c_{n-1}c_1}{c_n^2} - \frac{c_0}{c_n}\right) A + \frac{c_{n-1}c_0}{c_n^2} I. \quad (6.41)$$

Clearly this process can be continued. Thus any integer power of a matrix of order n can be reduced to a polynomial of the matrix, the highest degree of which is at most $n-1$. This fact can be used directly to obtain high powers of A.

Example 6.7.6. Find A^{100}, if $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$.

Solution 6.7.6. Since

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) = 0,$$

the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. The eigenvalues of A^{100} must be λ_1^{100} and λ_2^{100}, i.e.,

$$A^{100}\mathbf{x}_1 = \lambda_1^{100}\mathbf{x}_1, \quad A^{100}\mathbf{x}_2 = \lambda_2^{100}\mathbf{x}_2.$$

On the other hand, from the Cayley–Hamilton theorem, we know that A^{100} can be expressed as a linear combination of A and I, since A is of second order matrix ($n = 2$).

$$A^{100} = \alpha A + \beta I,$$

thus

$$\begin{aligned} A^{100}\mathbf{x}_1 &= (\alpha A + \beta I)\mathbf{x}_1 = (\alpha\lambda_1 + \beta)\mathbf{x}_1, \\ A^{100}\mathbf{x}_2 &= (\alpha A + \beta I)\mathbf{x}_2 = (\alpha\lambda_2 + \beta)\mathbf{x}_2. \end{aligned}$$

Therefore

$$\lambda_1^{100} = \alpha\lambda_1 + \beta, \quad \lambda_2^{100} = \alpha\lambda_2 + \beta.$$

It follows:

$$\begin{aligned} \alpha &= \frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1^{100} - \lambda_2^{100}\right) = \frac{1}{6}(4^{100} - 2^{100}), \\ \beta &= \frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1\lambda_2^{100} - \lambda_2\lambda_1^{100}\right) = \frac{1}{3}(4^{100} + 2^{101}). \end{aligned}$$

Hence

$$A^{100} = \frac{1}{6}(4^{100} - 2^{100})A + \frac{1}{3}(4^{100} + 2^{101})I.$$

Exercises

1. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 19 & 10 \\ -30 & -16 \end{pmatrix}.$$

Ans. $\lambda_1 = 4$, $\mathbf{x}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$; $\lambda_2 = -1$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

2. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 6-2\mathrm{i} & -1+3\mathrm{i} \\ 9+3\mathrm{i} & -4+3\mathrm{i} \end{pmatrix}.$$

Ans. $\lambda_1 = 2$, $\mathbf{x}_1 = \begin{pmatrix} 1-\mathrm{i} \\ 2 \end{pmatrix}$; $\lambda_2 = \mathrm{i}$, $\mathbf{x}_2 = \begin{pmatrix} 1-\mathrm{i} \\ 3 \end{pmatrix}$.

3. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 2 & -2 & 1 \\ 2 & -4 & 3 \\ 2 & -6 & 5 \end{pmatrix}.$$

Ans. $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda_3 = 2$, $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

4. If $U^\dagger U = I$, show that (a) the columns of U form an orthonormal set; (b) $UU^\dagger = I$ and the rows of U form an orthonormal set.

5. Show that eigenvalues of antihermitian matrix are either zero or pure imaginary.

6. Find the eigenvalues and eigenvectors of the following matrix:

$$\frac{1}{5}\begin{pmatrix} 7 & 6\mathrm{i} \\ -6\mathrm{i} & -2 \end{pmatrix},$$

and show explicitly that the two eigenvectors are orthogonal.

Ans. $\lambda_1 = 2$, $\mathbf{x}_1 = \begin{pmatrix} 2\mathrm{i} \\ 1 \end{pmatrix}$; $\lambda_2 = -1$, $x_2 = \begin{pmatrix} 1 \\ 2\mathrm{i} \end{pmatrix}$.

7. Show the two eigenvectors in the previous problem are orthogonal. Construct an unitary matrix U with the two normalized eigenvectors, and show that

$$U^\dagger U = I.$$

Ans. $U = \frac{1}{\sqrt{5}}\begin{pmatrix} 2\mathrm{i} & 1 \\ 1 & 2\mathrm{i} \end{pmatrix}$.

8. Find the eigenvalues and eigenvectors of the following symmetric matrix:

$$A = \frac{1}{5}\begin{pmatrix} 6 & 12 \\ 12 & -1 \end{pmatrix}.$$

Construct an orthogonal matrix U with the two normalized eigenvectors and show that

$$\widetilde{U}AU = \Lambda,$$

where Λ is a diagonal matrix whose elements are the eigenvalues of A.

Ans. $U = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$

9. Diagonalize the hermitian matrix

$$A = \begin{pmatrix} 1 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2 \end{pmatrix},$$

with a unitary similarity transformation

$$U^{\dagger}AU = \Lambda.$$

Find the unitary matrix U and the diagonal matrix Λ.

Ans. $U = \begin{pmatrix} -\frac{1+\mathrm{i}}{\sqrt{3}} & \frac{1+\mathrm{i}}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}.$

10. Diagonalize the symmetric matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

with a similarity transformation

$$U^{\dagger}AU = \Lambda.$$

Find the orthogonal matrix U and the diagonal matrix Λ.

Ans. $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$

11. Diagonalize the symmetric matrix

$$A = \frac{1}{3}\begin{pmatrix} -7 & 2 & 10 \\ 2 & 2 & -8 \\ 10 & -8 & -4 \end{pmatrix},$$

with a similarity transformation

$$\widetilde{U}AU = \Lambda.$$

Find the orthogonal matrix U and the diagonal matrix Λ.

Ans. $U = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$

12. If A is a symmetric matrix (so $\widetilde{A} = A$), S is an orthogonal matrix and $A' = S^{-1}AS$, show that A' is also symmetric.

13. If $\mathbf{u}$ and $\mathbf{v}$ are two column matrices in a two-dimensional space, and they are related by the equation

$$\mathbf{v} = C\mathbf{u},$$

where

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

find C^{-1} by the following methods:
(a) By Cramer's rule.
(b) Show that C is orthogonal, so $C^{-1} = \widetilde{C}$.
(c) The equation $\mathbf{v} = C\mathbf{u}$ means that C rotates $\mathbf{u}$ to $\mathbf{v}$. Since $\mathbf{u} = C^{-1}\mathbf{v}$, C^{-1} must rotate $\mathbf{v}$ back to $\mathbf{u}$. Therefore C^{-1} must also be a rotation matrix with an opposite direction of rotation.

14. Find the eigenvalues λ_1, λ_2, and the corresponding eigenvectors of the two-dimensional rotation matrix

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Find the unitary matrix U, such that

$$U^{\dagger}CU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Ans. $\mathrm{e}^{\mathrm{i}\theta}, \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}, \mathrm{e}^{-\mathrm{i}\theta}, \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}.$

15. Show that the canonical form (in which there is no crossproduct terms) of the quadratic expression

$$Q(x_1, x_2) = 7x_1^2 + 48x_1x_2 - 7x_2^2$$

is

$$Q'(x_1', x_2') = 25x_1^2 - 25x_2^2,$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}.$$

Find the orthogonal matrix S.

Ans. $S = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$.

16. If $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$ and $f(x) = x^3 + 3x^2 - 3x - 1$, find $f(A)$.

    Ans. $\begin{pmatrix} -13 & -26 \\ 13 & 26 \end{pmatrix}$.

17. If $A = \begin{pmatrix} 1 & 0 \\ 2 & \frac{1}{4} \end{pmatrix}$, find A^n and $\lim_{n\to\infty} A^n$.

    Ans. $A^n = \begin{pmatrix} 1 & 0 \\ \frac{8}{3} - \frac{8}{3}\left(\frac{1}{4}\right)^n & \left(\frac{1}{4}\right)^n \end{pmatrix}, \quad \lim_{n\to\infty} = \begin{pmatrix} 1 & 0 \\ \frac{8}{3} & 0 \end{pmatrix}$.

18. Solve for X, if $X^3 = \begin{pmatrix} -6 & 14 \\ -7 & 15 \end{pmatrix}$.

    Ans. $X = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$.

19. Solve the equation

$$M^2 - 5M + 3I = \begin{pmatrix} 1 & -4 \\ 2 & -5 \end{pmatrix}.$$

    Ans. $M_1 = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$, $M_2 = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$, $M_3 = \begin{pmatrix} 6 & -4 \\ 2 & 0 \end{pmatrix}$, $M_4 = \begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix}$.

20. According to Cayley–Hamilton theorem, every square matrix satisfies its own characteristic equation. Verify this theorem for each of the following matrices:

$$\text{(a)} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}.$$

21. Find A^{-1} by Cayley–Hamilton theorem if $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

    Ans. $A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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This bibliograph includes the references cited in the text and a few other books and table that might be useful.

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Index